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Invariants in dissipationless hydrodynamic media

By A. V. TUR[1] AND V. V. YANOVSKY[2]

[1] Laboratoire d'Energétique et de Mécanique Théorique et Appliquée, 2 Avenue de la Forêt de Haye, BP 160, 54504 Vandoeuvre Les Nancy, Cedex France

[2] Electro-Physical Scientifique Centre, The Ukrainian Academy of Sciences, Kharkov, Ukraine 310108

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We propose a general geometric method of derivation of invariant relations for hydrodynamic dissipationless media. New dynamic invariants are obtained. General relations between the following three types of invariants are established, valid in all models: Lagrangian invariants, frozen-in vector fields and frozen-in co-vector fields. It is shown that frozen-in integrals form a Lie algebra with respect to the commutator of the frozen fields. The relation between frozen-in integrals derived here can be considered as the Backlund transformation for hydrodynamic-type systems of equations. We derive an infinite family of integral invariants which have either dynamic or topological nature. In particular, we obtain a new type of topological invariant which arises in all hydrodynamic dissipationless models when the well-known Moffatt invariant vanishes.

1. Introduction

Conservation laws and topological properties play an important role in various models of continuous media. For instance the frozenness integral $\boldsymbol{J} = (\boldsymbol{\nabla} \times \boldsymbol{V})/\rho$ has been known from the time of Lord Kelvin while the frozenness of magnetic fields in magnetohydrodynamic (MHD) theory was discovered by Alfvén (1950). More complicated integrals which characterize linkage of vortex lines have been obtained by Moffatt (1969) (see also Moreau 1961; Frenkel 1982) and their topological meaning has been clarified by Arnol'd (1974). Similar helicity invariants have been obtained in MHD by Woltjer (1958) who also introduces the cross-helicity invariant (see e.g. Moffatt 1978; Berger & Field 1984). More complicated hydrodynamic models are exemplified by invariants for two-fluid plasma given by Sagdeev *et al.* (1986).

Among other types of invariants for hydrodynamic models, the following are also well known: the Lagrangian invariants, e.g. the Ertel invariant (Ertel 1942), and more complicated Hollmann invariants (Hollmann 1964); for two-fluid plasmas, such invariants have been derived by Sagdeev *et al.* (1986). These invariants have been successfully applied to quantitative and qualitative analysis of various phenomena. In particular, for incompressible fluid, the frozenness integral $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{V}$ plays an important role in the derivation of equations of motion for systems of point vortices and vortex systems by Novikov (1985) and for the derivation of exact vortex solutions (Batchelor 1967; Tur & Yanovsky 1984, 1991). Some integrals involving arbitrary functions are useful for stability analysis of various solutions along the lines proposed by Arnol'd (1969) and developed by Holm, Marsden & Weinstein (1985).

There naturally arises the question of the nature of the invariants and their relation to the structure of a particular physical model. One can present still more examples of local integrals and topological invariants of higher order with respect to the fields and their derivatives (see, for example, §§3, 5 and 6 below). A direct verification of these equations in coordinate representation however appears to be rather cumbersome and becomes practically impossible at still higher orders. Nevertheless there is a method not only to prove the existence of new local integrals and topological invariants, but also to express them explicitly in the coordinate representation. This method is the differential exterior forms calculus, which is presently widely used in physics (see, for example, Schultz 1982; Flanders 1989; von Westenholz 1981).

Recognizing the difficulties for the reader of a paper with a new unfamiliar mathematical language, we include a number of definitions of some basic concepts of differential geometry and clarify them when necessary. This background is essential because there is no other way to construct high-order invariants apart from that discussed here. In addition, it is very difficult to perceive relations between these invariants in a framework that does not use the topological language, and so topological fluid mechanics, currently the subject of intensive study, must also be applied here. A similar opinion was formulated by Moffatt (1990). 'My first assertion is that topological, rather than analytical techniques and language provide the natural framework for many aspects of fluid mechanical research that are now attracting intensive study'.

We propose here a general geometric method of derivation of invariant relations in hydrodynamic dissipationless media. New dynamic and topological invariants are obtained. General relations between the following three types of invariants are established to be valid in all models: Lagrangian invariants, frozen-in vector fields and frozen-in co-vector fields. Introduction of exterior forms allows to define, in a natural way, integral invariants for hydrodynamic media; some of them are of a dynamic nature, the others are topological. In particular, we obtain a new topological invariant which arises in all hydrodynamic dissipationless models when the well-known Moffatt invariant vanishes.

In Appendices A and B we list the simplest invariants of the first four generations for both incompressible and compressible fluids, which can be easily extended with the use of relations given here. Though the low-order invariants look familiar, the higher-order ones are rather cumbersome. Such invariants arise in all hydrodynamic models and they have a simple geometric interpretation connected with exterior differential forms.

Exterior forms of degree $p \leqslant 3$ exist in three-dimensional space, which leads to the existence of four types of local invariants (zero-forms or Lagrangian invariants, 1-forms or S-invariants, 2-forms associated with frozenness integrals, and 3-forms connected with the density ones). We obtain relations between different types of local invariants. These relations allow a large number of new conservation laws to be obtained. Some of these integrals are similar to the higher helicity integrals. The helicity integrals and the higher ones are also very important for understanding of some fine properties of the turbulent motion (Moffatt 1981; Levich, Shtilman & Tur 1991; Kiehn 1990, etc). Possible application of the new integrals will be presented elsewhere.

2. Local invariants

Let us consider the principal types of local invariant fields that appear in hydrodynamics. Here and below we understand the invariance not as just conservation of a certain quantity, but, more precisely, conservation of this quantity in the comoving reference frame. In other words, all temporal evolution of the considered field reduces to its advection. (Rigorous definitions will be presented after the introduction of some useful mathematical language of differential forms.) It is obvious that evolution of such an invariant field can be easily followed in Lagrangian coordinates (see (3.27)). This approach often provides useful information on the medium dynamics. However, it is much more important that the mere existence of local invariant fields leads to the existence of integral invariants and topological invariants which are conserved in the usual sense. Having thus found a sufficiently large set of local invariants, it is possible to construct new integral and topological invariants. We describe some types of local invariants below, making use of several well-known examples. We then demonstrate the impossibility of existence of other types of invariant fields in three-dimensional space.

Let us consider the system of hydrodynamical equations for incompressible fluid

$$\left.\begin{aligned} &\partial_t V+(V\cdot\nabla)\,V=-\nabla P,\\ &\operatorname{div} V=0. \end{aligned}\right\} \tag{2.1}$$

It is well-known that the quantity rot V is frozen into the fluid and satisfies the following equation:

$$\partial_t \operatorname{rot} V+(V\cdot\nabla)\operatorname{rot} V=(\operatorname{rot} V\cdot\nabla)\,V. \tag{2.2}$$

For another widely known example, one can take the MHD equations

$$\left.\begin{aligned} &\partial_t V+(V\cdot\nabla)\,V=\frac{-\nabla P}{\rho}+\frac{1}{4\pi\rho}\operatorname{rot}\boldsymbol{B}\times\boldsymbol{B},\\ &\partial_t \boldsymbol{B}+(V\cdot\nabla)\boldsymbol{B}=(\boldsymbol{B}\cdot\nabla)\,V,\\ &\quad\partial_t\rho+\operatorname{div}\rho V=0. \end{aligned}\right\} \tag{2.3}$$

In this model the field $\boldsymbol{B}/\rho$ is frozen into the medium (as easily follows from the system (2.3)) and obeys the equation

$$\partial_t\frac{\boldsymbol{B}}{\rho}+(V\cdot\nabla)\frac{\boldsymbol{B}}{\rho}=\left(\frac{\boldsymbol{B}}{\rho}\cdot\nabla\right)V. \tag{2.4}$$

A slightly more complicated example is connected with two-fluid plasma hydrodynamics (Sagdeev *et al.* 1986), which is described by the following system:

$$\left.\begin{aligned} &m_i n_i\frac{\mathrm{d}V_i}{\mathrm{d}t}=-\nabla P_i+en_i\left(\boldsymbol{E}+\frac{1}{c}[V_i\times\boldsymbol{B}]\right),\\ &m_e n_e\frac{\mathrm{d}V_e}{\mathrm{d}t}=-\nabla P_e-en_e\left(\boldsymbol{E}+\frac{1}{c}[V_e\times\boldsymbol{B}]\right),\\ &\frac{\partial n_e}{\partial t}+\operatorname{div}(n_e V_e)=0,\quad \frac{\partial n_i}{\partial t}+\operatorname{div}(n_i V_i)=0 \end{aligned}\right\} \tag{2.5}$$

and Maxwell's equations. Here V_i, n_i, m_i, P_i and V_e, n_e, m_e, P_e are the ion and electron velocity, density, mass and pressure, respectively. In this case the field that is frozen into the electron fluid is given by

$$J_e = \frac{\operatorname{rot} V_e - \dfrac{eH}{m_e c}}{n_e}, \quad \frac{\partial J_e}{\partial t} + (V_e \cdot \nabla J_e) = (J_e \cdot \nabla)\, V_e \tag{2.6}$$

and the field frozen into the ion fluid is

$$J_i = \frac{\operatorname{rot} V_i + \dfrac{eH}{m_i c}}{n_i}, \quad \frac{\partial J_i}{\partial t} + (V_i \cdot \nabla)\, J_i = (J_i \cdot \nabla)\, V_i. \tag{2.7}$$

A list of similar examples can be easily drawn up, but it is clearly seen that all the frozen in fields H/ρ, J_i, J_e, etc... are of the same form, while equations for the velocity field are significantly different. Thus, in various hydrodynamical models there exist vector fields frozen into the medium. We term the vector field J that obeys the equation

$$\partial_t J + (V \cdot \nabla)\, J = (J \cdot \nabla)\, V \tag{2.8}$$

as frozen into the medium, or the frozenness integral. It is important to note that this definition is independent of the type of the relevant velocity field equation, and thus it is universal for all hydrodynamic equations, although of course, expressions for the field J in terms of original fields in a given hydrodynamical system may differ from each other. For example, $J = \operatorname{rot} V$ in incompressible fluid, $J = \operatorname{rot} V/\rho$ in compressible fluid, $J = B/\rho$ in MHD, etc.

In geometrical terms, the definition (2.8) means that the field J line is advected by the medium (by the V field). In fact, it can be easily proved that the fields embedded in a continuous medium are governed by (2.8).

Let us consider such a field line parameterized by the variable s (e.g. the length measured along the line) in a continuous medium. Owing to the motions, $x = x(s, t)$, the field line moves. The vector field tangent to the line is defined as $J(x, t) = \mathrm{d}x(t, s)/\mathrm{d}s$, while the velocities of the points at the line coincide everywhere with the medium velocity, $\mathrm{d}x(s, t)/\mathrm{d}t = V(x, t)$. Differentiating $J(x, t)$ with respect to t, we easily obtain (2.8) for J. The integral curves for a given field $J(x, t)$ are defined through (Moffatt 1978)

$$J \times \mathrm{d}x = 0. \tag{2.9}$$

It can be easily seen that the pattern of integral curves of the field J is preserved after multiplication of J by a continuous non-vanishing function. This implies that when advection of field lines is accepted as the basis of definition of frozen-in fields, all fields of the type $f(x, t)\, J$ with $f(x, t) \neq 0$, are also frozen-in. Thus frozen-in fields can also obey equations of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = Y \cdot \nabla V + Y\left(\frac{\partial \ln f}{\partial t} + V \cdot \nabla \ln f\right), \tag{2.10}$$

where $Y = fJ$. This explains the differences in the forms of equations for frozen-in fields which can be met in literature. We emphasize that it is more natural from the physical viewpoint to define a frozen-in field based on the behaviour of its field lines (or integral trajectories). Thus we are interested in equivalence classes $\{J\}$ with respect to integral trajectories, rather than in individual field types (all frozen-in fields which have identical integral trajectories belong to the same class). We consider the fields governed by (2.8), rather than (2.10), as representatives of each class. The reason for this choice, which is given below, is associated with the fact that representation (2.8) is more natural.

Existence of the frozen-in fields rot $\boldsymbol{V}$, $\boldsymbol{H}/\rho$ and $\boldsymbol{J}$ in the examples above mentioned leads to theorems, similar to the Stokes theorem, on conservation of curl, field line circulation, etc. The importance of these theorems is widely acknowledged. Another consequence is the existence of topological invariants (Moffatt 1969; Tur & Yanovsky 1991). Note that new non-trivial frozen-in fields correspond to new higher-ordered integral and topological invariants.

Consider now another type of invariant met in hydrodynamics. The simplest of them are the so-called Lagrangian invariants, which are, by definition, the functions governed by the following equation:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + (\boldsymbol{V}\cdot\nabla)I = 0, \tag{2.11}$$

where $\boldsymbol{V}$ is the hydrodynamic velocity field. The physical meaning of Lagrangian invariants reduces to their advection by the flow. The definition (2.11), as well as (2.8), is naturally independent of the set of equations governing the velocity field. In this sense, this definition is universal for all hydrodynamic-type systems of equations.

To exemplify Lagrangian invariants one can consider the entropy density S and the Ertel invariant $I_{\mathrm{e}} = \mathrm{rot}\,\boldsymbol{V}\cdot\nabla S/\rho$ (Ertel 1942) (where ρ is the medium density). These invariants are obtained in compressible adiabatic hydrodynamics and are of great importance in geophysical hydrodynamics and dynamical meteorology. More complicated Lagrangian invariants can be obtained from the model (Hollmann 1964):

$$I_3 = \frac{\nabla S \times \nabla I_{\mathrm{e}}}{\rho}(\boldsymbol{V} - \nabla H), \quad I_4 = \frac{\nabla S \times \nabla I_3}{\rho}(\boldsymbol{V} - \nabla H), \quad I_S = \frac{\nabla S \times \nabla I_{\mathrm{e}}}{\rho}\nabla I_3. \tag{2.12}$$

Here $H = \int_0^t \Lambda\,\mathrm{d}t$ is the so-called action (with the Lagrange function $\Lambda = \frac{1}{2}V^2 - \eta$, where η is the entalpy per unit mass). New examples of Lagrangian invariants of some other hydrodynamic models are presented below.

The next type of local invariant, which is not so well known, is defined as a field $\boldsymbol{S}$, governed by the following equation:

$$\mathrm{d}\boldsymbol{S}/\mathrm{d}t = (\boldsymbol{S} \times \nabla) \times \boldsymbol{V}. \tag{2.13}$$

This equation probably needs some explanation since this representation is rarely met in the physical literature. Usually, this equation is represented in coordinate form, which makes it rather cumbersome. We give this form here for illustrative purposes (Batchelor 1967):

$$\frac{\partial S_j}{\partial t} + V^k \frac{\partial S_j}{\partial x^k} = S_m \frac{\partial V^m}{\partial x^j}. \tag{2.14}$$

The physical meaning of $\boldsymbol{S}$-invariants reduces to advection of the surfaces defined by $\boldsymbol{S}(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = 0$. In other words, the surface orthogonal to the vector field $\boldsymbol{S}$ is frozen into the flow. To be more precise, the equation $\boldsymbol{S}(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = 0$ defines a plane orthogonal to $\boldsymbol{S}$ at each position $\boldsymbol{x}$, so it defines a local field of planes frozen into the medium (see figure 1). If a global integral surface of the given field of planes exists

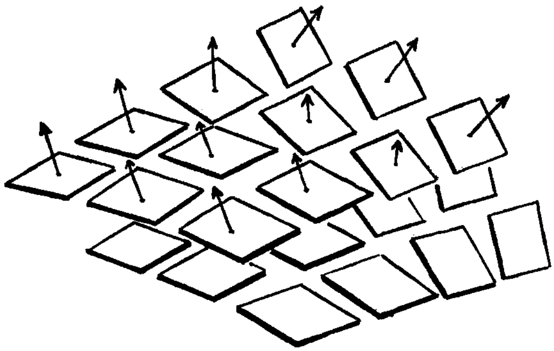


FIGURE 1. The field of planes defined by an S-type invariant. Some of the vectors S are marked by arrows. The left-hand side of the figure illustrates an integrable field of planes, the right-hand one shows a non-integrable field of planes.

(that is, the surface tangent to the field of planes at each position), then it is also frozen into the medium. However, in contrast to frozen-in vector fields, for which integral trajectories always exist, the field of planes has integral surfaces only if a special condition known as the Frobenius condition (Flanders 1989) is satisfied. In terms of the S-field, this condition can be easily represented as $S \operatorname{rot} S = 0$. It is easily verified that if this condition holds at the moment $t = 0$, then (as follows from (2.14)), it holds for any moment $t > 0$. This implies a consistency between the Frobenius condition and the dynamics of hydrodynamic media. It can be said that the existence of integral surfaces is a topological property independent of hydrodynamic motions. This easily follows from the fact that the proof of time conservation $S \operatorname{rot} S = 0$ is based only on the definition (2.14) of the S-invariant and does not involve a dynamical equation governing the velocity V of the medium.

Examples of invariants of the S-type are scarce, one such is the momentum of a small ring vortex (Roberts 1972; Kuzmin 1983). For a compressible fluid, the simplest S-invariant is the velocity field in a certain gauge, or $S = V - \nabla H$.

The mass conservation law is usually satisfied trivially. However, for some formal reasons, clarified in §3, it is convenient to consider the continuity equation,

$$\partial_t \rho + \operatorname{div}(\rho V) = 0 \tag{2.15}$$

associated with this law as the last type of locally invariant field.

We have thus defined four types of locally invariant quantities arising in hydrodynamic models of dissipationless media. These definitions are universal, in the sense that they are independent of the type of the hydrodynamic system considered. The following questions naturally arise in connection with the invariants discussed.

First, do the types mentioned include all the dynamic invariants in hydrodynamic models and what is their geometrical nature?

Second, since some local invariants are known in each hydrodynamic model (see the above examples), is it possible to build up some new invariants from the ones presented in the framework of these hydrodynamic models?

And, last, what do these invariants have in common with integral conservation laws, the topological ones in particular? Is it possible to construct new topological invariants in hydrodynamic models?

The answers to these questions are positive and we discuss them in the following sections.

3. Geometrical nature of local invariants and differential forms

Let us answer the simplest question: why there are only four types of local invariants?

A certain geometrical object is connected with each type of invariant presented above, namely differential form of degree p ($p = 0, 1, 2, 3$). The language of differential forms is widely used in the modern physical and mathematical literature (Arnol'd 1978; Flanders 1989: Schutz 1982; von Westenholz 1981), but for convenience we present some definitions and properties of differential forms here.

By definition, an external form of degree p or the p-form ω^p is a linear and skew-symmetric function of p vectors, that is

$$\omega(\alpha_1 \xi_1' + \alpha_2 \xi_1'', \xi_2, \dots, \xi_p) = \alpha_1 \omega(\xi_1', \xi_2, \dots, \xi_p) + \alpha_2 \omega(\xi_1'', \xi_2, \dots, \xi_p), \tag{3.1}$$

$$\omega(\xi_1, \dots, \xi_k, \xi_{k+1}, \dots, \xi_p) = (-1)\, \omega(\xi, \dots, \xi_{k+1}, \xi_k, \dots, \xi_p). \tag{3.2}$$

The set of all p-forms in R^n (R^n is the n-dimensional real linear space) makes up a linear space if one introduces addition and multiplication by a constant in the following manner:

$$\left.\begin{aligned} (\omega_1 + \omega_2)(\xi) &= \omega_1(\xi) + \omega_2(\xi), \\ \xi = (\xi_1, \dots, \xi_p); &\quad \xi_i \in \boldsymbol{R}^n, \\ (\alpha\omega)(\xi) &= \alpha\omega(\xi). \end{aligned}\right\} \tag{3.3}$$

In addition, an exterior product of forms $\omega_1^k \wedge \omega_2^l$ is defined on this set as resulting in a $(k+l)$-form. The exterior product is distributive, associative and anti-commutative:

$$\left.\begin{aligned} (\alpha_1 \omega_1^k + \alpha_2 \omega_2^k) \wedge \omega^l &= \alpha_1 \omega_1^k \wedge \omega^l + \alpha_2 \omega_2^k \wedge \omega^l, \\ (\omega^k \wedge \omega^l) \wedge \omega^m &= \omega^k \wedge (\omega^l \wedge \omega^m), \\ \omega^k \wedge \omega^l &= (-1)^{kl} \omega^l \wedge \omega^k. \end{aligned}\right\} \tag{3.4}$$

There is one more important operation: the interior product of forms is defined in the space of exterior forms. If ω^k is an exterior form, and V is a vector field, then the interior product associated with the k-form ω^{k-1} is defined as

$$i_V \omega^k(\xi_1, \dots, \xi_k) = \omega^k(V, \xi_1, \dots, \xi_{k-1}). \tag{3.5}$$

The interior product possesses a property that is usually called anti-differentiation:

$$i_V \omega_1^p \wedge \omega_2^l = (i_V \omega_1^p \wedge \omega_2^l + (-1)^p \omega_1^p \wedge (i_V \omega_2^l). \tag{3.6}$$

Moreover

$$i_{x+y}\omega = (i_x + i_y)\,\omega, \quad i_{fV}\omega = f i_V \omega. \tag{3.7}$$

Application of the interior product with the V-field to this 1-form $\omega^{(1)}$ gives

$$i_V \omega^{(1)}(\xi) = \omega^{(1)}(V).$$

Let us give some useful examples of exterior forms. For the simplest example of a 1-form we can take vectors coordinates in a given Cartesian frame $x_i = x_i(\xi)$. For another one can take the work of a force $\boldsymbol{F}$ for the displacement ξ:

$$\omega_F^1(\xi) = \boldsymbol{F} \cdot \xi. \tag{3.8}$$

Here $\boldsymbol{F}$ is a given vector, $\xi \in \boldsymbol{R}^n$.

For an example of a 2-form we can take the flux through the surface of a parallelogram formed by vectors ξ_1 and ξ_2:

$$\omega_V^2(\xi_1, \xi_2) = \boldsymbol{V} \cdot (\xi_1 \times \xi_2). \tag{3.9}$$

The volume of the parallelpiped with side ξ_1, ξ_2, and ξ_3 in R^3 is a 3-form:

$$\omega^3(\xi_1,\xi_2,\xi_3) = \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix} . \tag{3.10}$$

The exterior product is a generalization of the vector product. For example, one can easily prove in R^3 that (Arnol'd 1978)

$$\omega_A^1 \wedge \omega_B^1 = \omega_{A\times B}^2.$$

Consider now exterior differential forms. For this purpose we introduce a manifold which can be considered without loss of generality, as a hypersurface in $\boldsymbol{R}^n$, where n is sufficiently large. At each point of the manifold there exists a set of vectors tangent to the surface and forming a linear vector space denoted as $\boldsymbol{TM}$. The union $\boldsymbol{TM} = \bigcup_{X\in M} \boldsymbol{TM}_x$ is called the tangent foliation of the manifold $\boldsymbol{M}$. Then, by definition, one calls the differential p-form $\omega^p|_x$ at a point x of a manifold $\boldsymbol{M}$ the external p-form on the tangent vectors belonging to $\boldsymbol{TM}_x$. If such a form is defined at each point of the manifold $\boldsymbol{M}$ and it is differential, then a p-form is defined on the manifold $\boldsymbol{M}$.

For the simplest example of a differential 1-form one can take a differential of a function. Let us take for the manifold $\boldsymbol{M}$ a linear space with coordinates $x_1,\ldots,x_n$. The components $\xi_1,\ldots,\xi_n$ of a tangent vector $\xi\in \boldsymbol{TR}_x^n$ are the values of the differentials of the coordinates $\mathrm{d}x_1,\ldots,\mathrm{d}x_n$ on the vector ξ. Any differential 1-form in R^n with a given reference frame $x_1,\ldots,x_n$ can be uniquely represented as

$$\omega = a_1(\boldsymbol{x})\,\mathrm{d}x_1 + \ldots + a_n(\boldsymbol{x})\,\mathrm{d}x_n, \tag{3.11}$$

where $a_i(\boldsymbol{x})$ are smooth functions. In addition the exterior product of the basic forms $\mathrm{d}x_i$ forms a basis in the space of exterior differential forms. It can be proved that each differential k-form in $\boldsymbol{R}^n$ with a chosen frame $x_1,\ldots,x_n$ is uniquely represented as (see e.g. Arnol'd 1978)

$$\omega^k = \sum_{i_1<\ldots<i_k} a_{i_1,\ldots,i_k}(\boldsymbol{x})\mathrm{d}x_{i_1} \wedge,\ldots,\wedge\, \mathrm{d}x_{i_k}. \tag{3.12}$$

The 0-form is just a smooth function of $\boldsymbol{R}^n$. Let us now describe the differentiation operation in the form space. The exterior differentiation d transforms k-forms into $(k+1)$-forms and is characterized by the following properties:

$$\left.\begin{aligned} \mathrm{d}(\alpha\omega_1+\beta\omega_2) &= \alpha\,\mathrm{d}\omega_1+\beta\,\mathrm{d}\omega_2, \\ \mathrm{d}(\omega^l \wedge \omega^k) &= (\mathrm{d}\omega^l) \wedge \omega^k + (-1)^l\omega^l \wedge \mathrm{d}\omega^k, \\ \mathrm{d}(\mathrm{d}\omega) &\equiv 0. \end{aligned}\right\} \tag{3.13}$$

In the case of $\boldsymbol{R}^n$, the exterior differentiation operation is defined as

$$\mathrm{d}\omega^k = \sum \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge,\ldots,\mathrm{d}x_{i_k}. \tag{3.14}$$

Let us define another differential operator which is very important for our purposes. The Lie derivative L_V along a vector field V is understood as the operator which transforms ω^k into ω^k as according to

$$L_V\omega = (i_V\mathrm{d} + \mathrm{d}i_V)\,\omega. \tag{3.15}$$

Such a derivative is defined for all tensors fields, not only for differential forms.

The Lie derivative has several typical properties (Flanders 1989; Schutz 1982) which can be easily proved. It is commutative with the operation of external differentiation,

$$\mathrm{d}L_V = L_V \mathrm{d}.$$

Moreover,
$$L_V i_y - i_y L_V = i_{[V,y]}, \quad L_V L_y - L_y L_V = L_{[V,y]}, \tag{3.16}$$

where the square brackets denote a commutator of the vector fields:

$$[V, Y] = V_e \frac{\partial Y_i}{\partial x_e} - Y_e \frac{\partial V_i}{\partial x_e}.$$

Let us now introduce the definition of local dynamic invariants using the language of differential forms, considering the time as a parameter.

We call invariant the differential form which satisfies the following equation:

$$\partial_t \omega + L_V \omega = 0, \tag{3.17}$$

where V is the velocity field vector.

We restrict ourselves to the case common for hydrodynamic models, the three-dimensional space $\boldsymbol{R}^3$. Then it is easy to prove that only differential forms ω^p for $p = 0, 1, 2, 3$ are non-vanishing. The reason for vanishing of the form of any higher degree is associated with the form antisymmetry and with the linear dependence of any four vectors in $\boldsymbol{R}^3$, that is

$$\xi_4 = \alpha\xi_1 + \beta\xi_2 + \gamma\xi_3.$$

Therefore

$$\begin{aligned}\omega^4(\xi_1, \xi_2, \xi_3, \xi_4) &\equiv \omega^4(\xi_1, \xi_2, \xi_3, \alpha\xi_1 + \beta\xi_2 + \gamma\xi_3) \\ &\equiv \alpha\omega^4(\xi_1, \xi_2, \xi_3, \xi_1) + \beta\omega^4(\xi_1, \xi_2, \xi_3, \xi_2) + \gamma\omega^4(\xi_1, \xi_2, \xi_3, \xi_3).\end{aligned}$$

Then it is easy to prove that the forms containing two coinciding arguments vanish because of their skew symmetry (see (3.4)). That is why $\omega^4 \equiv 0$ in $\boldsymbol{R}^3$. Hence there exist only four types of dynamic invariants associated with invariant differential forms of degree p for $p = 0, 1, 2, 3$, respectively.

Consider now how the definition (3.17) is connected with the usual definitions of local dynamic invariants in hydrodynamic models. Let us start with 0-forms which are the usual functions, $I = I(t, x, y, z)$. Application of the Lie derivative L_v to a function reduces it to the derivative along the vector field V, and the definition (3.17) for the 0-forms becomes

$$\frac{\partial I}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) I = 0. \tag{3.18}$$

Comparing this equation with the standard definition of Lagrangian invariants (2.8), one easily finds that 0-forms are Lagrangian invariants.

Let us discuss now 1-forms, which can always be presented in the coordinate form in $\boldsymbol{R}^3$ as (see (3.11))

$$\omega^1 = \sum_{i=1}^{3} S_i(t, \boldsymbol{x})\, \mathrm{d}x_i. \tag{3.19}$$

The coordinate form of (3.17) for 1-forms (3.19) reduces, after explicit evaluation of the Lie derivative, to

$$\left(\partial_t S_i + V^k \frac{\partial S_i}{\partial x^k} + S_k \frac{\partial V^k}{\partial x_i}\right) \mathrm{d}x^i = 0. \tag{3.20}$$

This implies that an invariant 1-form determines an S-invariant (see (2.11)). It should be noted that this fact is already important since, for example, it enables one to obtain an exact solution of (2.11) in Lagrangian coordinates. In fact, let at $t = 0$, $S_i = S_{0i}(\boldsymbol{x}_0, t)$, where $\boldsymbol{x}_0$ are Lagrangian coordinates. Consider the trajectories of Lagrangian particles $x_i = x_i(\boldsymbol{x}_0, t)$ as a transformation of variables. Since an exterior form is invariant with respect to transformation of variables, we have the following identity:

$$S_i(\boldsymbol{x}^0, t)\frac{\partial x^i}{\partial x_j^0}\mathrm{d}x_j^0 = = S_{0j}\,\mathrm{d}x_j^0;$$

whence it follows that

$$S_i(\boldsymbol{x}_0, t) = S_{0j}(\boldsymbol{x}_0)\frac{\partial x_j^0}{\partial x_i} \tag{3.21}$$

is an exact solution of (2.11) in Lagrangian variables. The validity of (3.21) can be verified by a direct substitution into (2.11).

Consider now to the 3-form, which in $\boldsymbol{R}^3$ is

$$\omega^3 = f(t; \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_1 \wedge \mathrm{d}\boldsymbol{x}_2 \wedge \mathrm{d}\boldsymbol{x}_3. \tag{3.22}$$

The coordinate representation of (3.17) for the 3-form (3.22) reduces to the following equation:

$$\left(\partial_t f + \frac{\partial}{\partial x_i}(V_i f)\right)\mathrm{d}\boldsymbol{x}_1 \wedge \mathrm{d}\boldsymbol{x}_2 \wedge \mathrm{d}\boldsymbol{x}_3 = 0. \tag{3.23}$$

It is easy to recognize the continuity equation in this. Thus, invariant 3-forms denote invariant densities.

We emphasize the existence of an invariant 3-form, the mass form

$$\omega^3 = \rho\,\mathrm{d}\boldsymbol{x}_1 \wedge \mathrm{d}\boldsymbol{x}_2 \wedge \mathrm{d}\boldsymbol{x}_3$$

(where ρ is the density), which arises in all hydrodynamic models. From the physical viewpoint, this form is associated with the property of the continuity of hydrodynamic media that distinguishes hydrodynamic models as a separate class of physical systems.

Consider now the invariant 2-form which can be represented in the given reference frame as

$$\omega_A^2 = A_1\,\mathrm{d}\boldsymbol{x}_2 \wedge \mathrm{d}\boldsymbol{x}_3 + A_2\,\mathrm{d}\boldsymbol{x}_3 \wedge \mathrm{d}\boldsymbol{x}_1 + A_3\,\mathrm{d}\boldsymbol{x}_1 \wedge \mathrm{d}\boldsymbol{x}_2. \tag{3.24}$$

The coordinate form of (3.17) for the 2-form (3.24) leads to the equation

$$\partial_t \boldsymbol{A} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}\,\mathrm{div}\,\boldsymbol{V} = (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{V}. \tag{3.25}$$

Comparing this equation and the definition of the frozen-in integral (2.4) one can easily see that, after introduction of the field $\boldsymbol{J} \equiv \boldsymbol{A}/\rho$, equation (3.25) coincides with the definition (2.4). This mean that an invariant 2-form defines a frozen-in integral. It is easy to introduce a definition for the frozen-in integral in the coordinate form using the special invariant 3-form of mass:

$$\omega_\rho^3 \equiv \rho\,\mathrm{d}\boldsymbol{x}_1 \wedge \mathrm{d}\boldsymbol{x}_2 \wedge \mathrm{d}\boldsymbol{x}_3.$$

The vector field $\boldsymbol{J}$ is frozen into the medium if $\omega^2 = i_J\omega_\rho^3$ is an invariant 2-form (that is, it is governed by (3.17)). The field $\boldsymbol{J}$ satisfies the following equation:

$$\partial_t \boldsymbol{J} + L_V \boldsymbol{J} = 0, \tag{3.26}$$

which coincides with (2.4). One can easily verify that (3.26) coincides with (2.4), because the Lie derivative of vector fields coincides with their commutator, $L_V J = [V, J]$, and has the following coordinate representation on the manifold (see, for example, Arnol'd 1978; Flanders 1989; von Westenholz 1981):

$$[\boldsymbol{A}, \boldsymbol{B}] = A^k \frac{\partial B^i}{\partial x^k} - B^k \frac{\partial A^i}{\partial x^k}. \tag{3.27}$$

Thus, the evolution of the frozen-in vector fields reduces to their advection along the streamlines. This fact is the most important feature of the frozen-in fields, or their geometric meaning. This interpretation allows one to solve (2.9) easily in Lagrangian coordinates:

$$J^i(\boldsymbol{x}_0, t) = J^k_0(\boldsymbol{x}_0) \frac{\partial x^i}{\partial x^k_0}.$$

This solution is a trivial transformation of a vector field $\boldsymbol{J}$ under transformation of coordinates due to the mapping $\boldsymbol{x}_0 \to \boldsymbol{x}$. This solution has been well known since the time of Euler and has been obtained by direct integration of (2.1).

Some consequences and advantages of the proposed interpretation of local dynamical invariants can be summarized as follows.

(i) Local dynamical invariants are geometric objects, namely invariant differential forms.

(ii) There exist only four types of local invariants corresponding to p-forms ($p = 0, 1, 2, 3$). This fact is a direct consequence of the three-dimensional nature of the space considered in hydrodynamic problems.

(iii) All four kinds of equations (2.4), (2.8), (2.11), (2.12), which define local invariants, reduce to a unique universal equation (3.17) defining an invariant differential form.

(iv) The exact solutions of equations governing local invariants in Lagrangian variables, having the forms

$$\begin{aligned} I(\boldsymbol{x}_0, t) &= I_0(\boldsymbol{x}_0), \\ S_i(\boldsymbol{x}_0, t) &= S_{0j}(\boldsymbol{x}_0)\, \partial x^j_0 / \partial x_i, \\ J^i(\boldsymbol{x}_0, t) &= J^j_0(\boldsymbol{x}_0)\, \partial x^i / \partial x^j_0, \\ \rho(\boldsymbol{x}_0, t) &= \rho_0(\boldsymbol{x}_0) \det(\partial x_0 / \partial x) \end{aligned}$$

are obvious consequences of the invariance of the corresponding differential forms.

4. Relations among different types of local invariants (new conservation laws)

The above examples demonstrate that in each hydrodynamic model a few local invariants are known (see examples in §2). The following question is thus important: how to construct new dynamic invariants based on a certain number of the known ones?

In this section we obtain a universal relation which enables us to construct new invariants in any hydrodynamic model. The possibility of construction of such universal relations, valuable for all hydrodynamic models, is based on the definition of the local invariants in terms of differential forms. These definitions (for example (3.17)) do not depend on hydrodynamic equations for velocity fields. This is the only reason for universality of the relations among the invariants independent of equations for the velocity field.

Let us proceed to a derivation of universal relations for local hydrodynamic invariants. Consider an invariant differential form ω^p, governed by the equation

$$\partial_t \omega^p + L_V \omega^p = 0.$$

Since the exterior derivative is commutative with the Lie derivative (that is, $\mathrm{d}L_V = L_V \mathrm{d}$), we apply an exterior differential to this equation to obtain

$$\partial_t(\mathrm{d}\omega^p) + L_V(\mathrm{d}\omega^p) = 0.$$

Hence, by definition, the form $\omega^{p+1} = \mathrm{d}\omega^p$ is an invariant $(p+1)$-form. This means that we have obtained three relations among the local invariants, namely

$$\omega^{p+1} = \mathrm{d}\omega^p \quad (p = 0, 1, 2, 3). \tag{4.1}$$

The coordinate representations reduce to the following:

$$\boldsymbol{S}' = \nabla I, \tag{4.2}$$

$$\rho \boldsymbol{J}' = \operatorname{rot} \boldsymbol{S}, \tag{4.3}$$

$$\rho' = \operatorname{div}(\rho \boldsymbol{J}). \tag{4.4}$$

It follows that if in a hydrodynamic system of equations one knows a Lagrangian invariant $I(\boldsymbol{x}, t)$, or $\boldsymbol{S}$-invariant, or a field $\boldsymbol{J}(\boldsymbol{x}, t)$, then using these relations one can construct a new field $\boldsymbol{S}'(\boldsymbol{x}, t)$, which is $\boldsymbol{S}$-invariant, or a new frozen-in field $\boldsymbol{J}'$, or a new density ρ'. Relations (4.2)–(4.4) also can be derived directly in the coordinate form, using one of the equations (2.8), (2.11), (2.14), or (2.15); however, the derivation in terms of differential forms presented above is the most simple and short. Let us present some simple examples of applications of these relations.

In compressible adiabatic fluid the entropy density $S(\boldsymbol{x}, t)$ is a Lagrangian invariant (see e.g. Batchelor 1967).

An $\boldsymbol{S}$ invariant follows from (4.2) as

$$\boldsymbol{S} = \nabla S(\boldsymbol{x}, t). \tag{4.5}$$

Using the Ertel invariant (see §2) one can obtain another $\boldsymbol{S}$-invariant for the same system:

$$\boldsymbol{S}' = \nabla[\operatorname{rot} \boldsymbol{V} \cdot \nabla S/\rho]. \tag{4.6}$$

In the same hydrodynamic model there exists an $\boldsymbol{S}$-invariant $\boldsymbol{S} = \boldsymbol{V} - \nabla H$ (see the example in §2), which can be used as the basic one. Then relation (4.3) leads to the well-known frozen-in integral

$$\boldsymbol{J} = \operatorname{rot} \boldsymbol{S}/\rho \equiv \operatorname{rot} \boldsymbol{V}/\rho.$$

When the frozen-in field is used for construction of an invariant density according to (4.4), it cannot lead directly to a non-trivial result since $\rho' = 0$. However, taking into account that a frozen-in integral preserves this property, being multiplied by a Lagrangian invariant (see (2.10)), it is convenient to use the following starting form:

$$\boldsymbol{J} = (S/\rho) \operatorname{rot} \boldsymbol{V}. \tag{4.7}$$

Then relation (4.4) yields the following new invariant density:

$$\rho' = \operatorname{div}(S \operatorname{rot} \boldsymbol{V}). \tag{4.8}$$

These typical examples illustrate the possibility of deriving new conservation laws using the general theorem on invariant forms, $\omega^{p+1} = \mathrm{d}\omega^p$. The number of such examples can be easily extended in the framework of any particular model including the hydrodynamic one considered (see the Appendices).

Consider now some relations which will be useful below. We assume that a hydrodynamic model has two invariants forms, ω_1^k and ω_2^l. Then the $(k+l)$-form $\omega_1^k \wedge \omega_2^l$ is also invariant. Indeed, we have by definition

$$\partial_t \omega_2^l + L_V \omega_2^l = 0, \quad \partial_t \omega_1^k + L_V \omega_1^k = 0.$$

Multiplying the first equation from the left by ω_1^k and the second one from the right by ω_2^l and summing them we obtain

$$\omega_1^k \wedge (\partial_t \omega_2^l) + (\partial_t \omega_1^k) \wedge \omega_2^l + \omega_1^k \wedge (L_V \omega_2^l) + (L_V \omega_1^k) \wedge \omega_2^l = 0.$$

Taking into account linearity of the operators ∂_t and L_V we have

$$\partial_t(\omega_1^k \wedge \omega_2^l) + L_V(\omega_1^k \wedge \omega_2^l) = 0.$$

Hence, the form $\omega^{k+l} = \omega_1^k \wedge \omega_2^l$ is an invariant one from the definition of a $(k+l)$-form:

$$\omega^{k+l} = \omega_1^k \wedge \omega_2^l. \tag{4.9}$$

A coordinate representation of this relation for $k = 0$ and $l = 0, 1, 2, 3$, implies that multiplication by a Lagrangian invariant does not violate its invariance, i.e.

$$I' = I_1(\boldsymbol{x}, t) I_2(\boldsymbol{x}, t), \boldsymbol{S}' = I_1(\boldsymbol{x}, t)\, \boldsymbol{S}(\boldsymbol{x}, t), \boldsymbol{J}' = I_1(\boldsymbol{x}, t)\, \boldsymbol{J}(\boldsymbol{x}, t), \rho' = I_1(\boldsymbol{x}, t)\, \rho(\boldsymbol{x}, t). \tag{4.10}$$

At $k = l = 1$ one obtains

$$\rho \boldsymbol{J}' = \boldsymbol{S}_1(\boldsymbol{x}, t) \times \boldsymbol{S}_2(\boldsymbol{x}, t). \tag{4.11}$$

Thus, given any two $\boldsymbol{S}$-invariants it is possible to obtain a frozen-in integral. When $k = 1$ and $l = 2$, we have

$$\rho' = \rho \boldsymbol{J}(\boldsymbol{x}, t) \cdot \boldsymbol{S}(\boldsymbol{x}, t). \tag{4.12}$$

The new local dynamic invariants in (4.10)–(4.12), obtained from the starting set of local dynamic invariants, are indicated by the prime.

Equations (4.10)–(4.12) can also be used for constructing invariants from any starting set. For example, the use of $\boldsymbol{S}$-invariants (4.5) and (4.6) for compressible fluid (in the adiabatic case) leads to a new frozen-in integral given by

$$\boldsymbol{J}'' = \frac{1}{\rho} \nabla S \times \nabla\, (\operatorname{rot} \boldsymbol{V} \cdot \nabla S / \rho), \tag{4.13}$$

where (4.11) has been used.

One more frozen-in integral follows from relation (4.11) when one chooses $\boldsymbol{S}_1 = \nabla S$ and $\boldsymbol{S}_2 = \boldsymbol{V} - \nabla H$ for the $\boldsymbol{S}$-invariants:

$$\boldsymbol{J}''' = \frac{1}{\rho} \nabla S \times (\boldsymbol{V} - \nabla H). \tag{4.14}$$

Starting from the frozen-in fields $\boldsymbol{J} = \operatorname{rot} \boldsymbol{V}/\rho$ and $\boldsymbol{S}_1 = \nabla S$, (3.5), one can easily construct examples of invariant densities. Equation (4.12) implies that

$$\rho' = \nabla S \operatorname{rot} \boldsymbol{V}. \tag{4.15}$$

When one takes $\boldsymbol{J}''$, (4.13), as a frozen-in field and $\boldsymbol{S}_2 = \boldsymbol{V} - \nabla H$ for an $\boldsymbol{S}$-invariant, (4.11) yields a new invariant density

$$\rho'' = (\boldsymbol{V} - \nabla H) \left(\nabla S \times \nabla \left(\frac{\operatorname{rot} \boldsymbol{V} \cdot \nabla S}{\rho} \right) \right). \tag{4.16}$$

The equations (4.15) and (4.16) satisfy the continuity equation and have the allied integral invariants, to be discussed in the next section in more detail. Of course, the newly obtained invariants, when used instead of the starting ones, allow one to deduce many further invariants for a given model with the help of (4.2)–(4.4) and (4.10)–(4.12).

The above relations transform a starting invariant form into a higher-degree one. Let us discuss now the degree-reducing relations that transform an invariant p-form into an invariant $(p-1)$-form. Such relations follow from the existence of the interior product operation. However, the interior product of an invariant exterior differential form and an arbitrary vector field violates its invariance.

Let us prove that the interior product of an invariant differential p-form and a frozen-in vector field represents an invariant $(p-1)$-form.

To prove this we use two relations. The first one is a canonical commutation rule of the Lie derivative L_X and the interior product operation i_Y (see §3):

$$L_X i_Y - i_Y L_X = i_{[X, Y]}.$$

In this expression the commutator of vector fields is denoted by square brackets, see (3.27). The second one is the commutation rule of the partial derivative $\partial/\partial t = \partial_t$ and the interior product i_Y:

$$\partial_t i_Y - i_Y \partial_t = i_{\partial_t Y}.$$

The proof of this relation is trivial. Applying, from the left, the interior product operation with the frozen-in field i_J to the equation for an invariant form, we have

$$i_J \partial_t \omega + i_J L_V \omega = 0.$$

Using the commutation rules and $i_A + i_B = i_{A+B}$ (that is, the linearity of the interior product), transform this equation to

$$\partial_t(i_J \omega^p) + L_V(i_J \omega^p) - i_{\partial_t J + [V, J]} \omega^p = 0.$$

The last term on the left-hand side vanishes since, by definition, $\partial_t \boldsymbol{J} + [\boldsymbol{V}, \boldsymbol{J}] = 0$ for a frozen-in field (see (3.26), (3.27)), hence

$$\partial_t(i_J \omega^p) + L_V(i_J \omega^p) = 0.$$

So, we have proved that the form

$$\omega^{p-1} = i_J \omega^p \tag{4.17}$$

is an invariant $(p-1)$-form, if ω^p is an invariant form and $\boldsymbol{J}$ is a frozen-in vector field.

In coordinate representation, formula (4.17) gives the following two relations between different invariants (the third one is trivial and coincides with the definition of the frozen-in field $\omega^2 = i_J \omega^3_\rho$):

$$I' = \boldsymbol{J} \cdot \boldsymbol{S}, \tag{4.18}$$

$$\boldsymbol{S}' = \rho[\boldsymbol{J} \times \boldsymbol{J}_1]. \tag{4.19}$$

Let us give some examples of applications of these relations. In a compressible adiabatic fluid we take for the starting set the frozen-in integral $\boldsymbol{J} = \operatorname{rot} \boldsymbol{V}/\rho$ and the S-invariants $\boldsymbol{S}_1 = \nabla S$ and $\boldsymbol{S}_2 = \boldsymbol{V} - \nabla H$. Relation (4.18) gives the well-known Ertel invariant (Ertel 1942) and a new Lagrangian invariant,

$$I_e = \frac{\operatorname{rot} \boldsymbol{V} \cdot \nabla S}{\rho}, \quad I = \frac{\operatorname{rot} \boldsymbol{V} \cdot (\boldsymbol{V} - \nabla H)}{\rho}.$$

It is easy to notice that if one takes for the frozen-in integrals $\boldsymbol{J}$, (4.13), and $\boldsymbol{S}_2 = \boldsymbol{V}-\nabla H$, one obtains from (4.17) the Hollman invariant (Hollman 1964) (2.12),

$$I_3 = \frac{\nabla S \times \nabla I_{\mathrm{e}}}{\rho} \cdot (\boldsymbol{V}-\nabla H).$$

The rest of invariants (2.12) and some new ones are also easily obtained with the subsequent use of the above-mentioned general relations.

Taking as the frozen-in integrals $\boldsymbol{J} = \operatorname{rot} \boldsymbol{V}/\rho$ and $\boldsymbol{J}$ obtained from (4.13) or (4.14) and substituting them into (4.19) we have the following $\boldsymbol{S}$-invariants:

$$\boldsymbol{S}' = \frac{1}{\rho}\left(\operatorname{rot} \boldsymbol{V} \times \left(\nabla S \times \nabla\left(\frac{\operatorname{rot} \boldsymbol{V} \cdot \nabla S}{\rho}\right)\right)\right), \tag{4.20}$$

$$\boldsymbol{S}'' = \frac{1}{\rho}(\operatorname{rot} \boldsymbol{V} \times (\nabla S \times (\boldsymbol{V}-\nabla H))). \tag{4.21}$$

Let us now approach the remaining class of relations which associate the new invariant forms of the same degree to the invariant starting forms. The existence of that kind of relation is due to the Lie derivative along a vector field. It is well known that the Lie derivative transforms a p-form into a p-form (Arnol'd 1978; Flanders 1989; Schutz 1982). However, if one evaluates the Lie derivative of an invariant form along an arbitrary vector field, the acquired form may not be invariant.

It is easy to prove that, in fact, the Lie derivative along a frozen-in vector field transforms an invariant form into an invariant one, that is

$$\omega_1^p = L_J \omega^p \tag{4.22}$$

is an invariant p-form if $\boldsymbol{J}$ is a frozen-in vector field and ω^p is an invariant form.

Strictly speaking, (4.22) is a simple consequence of the previously obtained formulae (4.1) and (4.17). Nevertheless, it leads to a number of useful relations. One can easily become convinced of this by noticing that the Lie derivative can be expressed in terms of the operators of exterior differentiation and interior product as $L_J = \mathrm{d}i_J + i_J\mathrm{d}$ (see e.g. Arnol'd 1978; Flanders 1989; Schutz 1982). The correspondence with the previously proved relations becomes evident.

In coordinate representation it follows from (4.22) that

$$I' = \boldsymbol{J} \cdot \nabla I, \tag{4.23}$$

$$S_i = J_m \frac{\partial S_i}{\partial x_m} + S_m \frac{\partial J_m}{\partial x_i}, \tag{4.24}$$

$$\boldsymbol{J}' = [\boldsymbol{J}_1, \boldsymbol{J}_2], \tag{4.25}$$

$$\boldsymbol{J}'' = \frac{1}{\rho''}(\operatorname{div} \boldsymbol{J}_1 \rho)\, \boldsymbol{J}, \tag{4.26}$$

$$\rho' = \operatorname{div} \boldsymbol{J}_1 \rho. \tag{4.27}$$

In these formulae the starting invariants are denoted by the corresponding letters I, $\boldsymbol{S}$, $\boldsymbol{J}$, ρ, and invariants of the new generation are distinguished by primes. Moreover, in these relations ρ, ρ' and ρ'' are not necessarily the density of the medium since for ρ, ρ' and $\rho\infty$ one can take any invariant density (for example, (4.8), (4.15) and (4.16)).

Let us make some particular remarks on the expression (4.25), $\boldsymbol{J}_3 = [\boldsymbol{J}_1, \boldsymbol{J}_2]$. This important property endows the frozen-in vector field with the Lie algebra structure.

In fact, we have proved that the commutator of two frozen-in vector fields is also a frozen-in vector field, and, as follows from the commutator definition (3.27), these fields anti-commute and satisfy the Jacoby identity:

$$[\boldsymbol{J}_1, \boldsymbol{J}_2] = -[\boldsymbol{J}_2, \boldsymbol{J}_1],$$
$$[[\boldsymbol{J}_1, \boldsymbol{J}_2], \boldsymbol{J}_3] + [[\boldsymbol{J}_3, \boldsymbol{J}_1], \boldsymbol{J}_2] + [[\boldsymbol{J}_2, \boldsymbol{J}_3], \boldsymbol{J}_1] = 0.$$

Thus, the frozen-in vector integrals form a Lie algebra with the multiplication operation defined by the commutator of the vector fields.

It is possible to use (4.25) for construction of new frozen-in vector fields in accordance with the previously given set $(\boldsymbol{J}_1, \boldsymbol{J}_2)$.

Note that an infinite-dimensional Lie algebra of vector fields possesses folia enumerable by the velocity field. The vector fields that belong to a given folio form a Lie subalgebra of frozen-in fields. This remark is to be of importance when deriving exact solutions in hydrodynamic media.

Let us assume that we have two exact solutions of a hydrodynamic system of equations:

$$\boldsymbol{J}_1 = \boldsymbol{J}_1(\boldsymbol{x}, t), \quad \boldsymbol{V}_1 = \boldsymbol{V}(\boldsymbol{x}, t), \quad \text{and} \quad \boldsymbol{J}_2 = \boldsymbol{J}_2(\boldsymbol{x}, t), \quad \boldsymbol{V}_2 = \boldsymbol{V}(\boldsymbol{x}, t).$$

Here the indices enumerate different exact solutions, Then, with the help of (4.25), one can construct a new solution

$$\boldsymbol{V}_3 = \boldsymbol{V}(\boldsymbol{x}, t), \quad \boldsymbol{J}_3 = [\boldsymbol{J}_1(\boldsymbol{x}, t), \boldsymbol{J}_2(\boldsymbol{x}, t)].$$

So, in this case (4.25) looks like the Backlund transformation (see e.g. Newel 1985) that transforms solutions of a hydrodynamic system of equations into its new solutions.

Let us now give some examples of how to use the relations obtained between the invariants (4.23)–(4.27) to construct new invariants in compressible ideal fluid. Taking as the starting ones $\boldsymbol{J}_1 = \operatorname{rot} \boldsymbol{V}/\rho$ and $\boldsymbol{J} = (1/\rho)\,\nabla S \times \nabla \cdot (\operatorname{rot} \boldsymbol{V} \cdot \nabla S/\rho)$ and using (4.25) we have

$$\boldsymbol{J} = \operatorname{rot}\left(\frac{\operatorname{rot} \boldsymbol{V}}{\rho} \times \frac{1}{\rho}\left(\nabla S \times \nabla\left(\frac{\operatorname{rot} \boldsymbol{V} \cdot \nabla S}{\rho}\right)\right)\right). \tag{4.28}$$

This and the previous examples have been given for compressible fluid and naturally do not exhaust all possible cases. Let us now present invariants of ideal MHD, that has the following hydrodynamic system of equations:

$$\left.\begin{aligned}
&\rho(\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla \boldsymbol{V}) = -\nabla P + \frac{1}{4\pi}(\operatorname{rot} \boldsymbol{B} \times \boldsymbol{B}),\\
&\partial_t \rho + \operatorname{div}(\rho \boldsymbol{V}) = 0,\\
&\partial_t S + (\boldsymbol{V} \cdot \nabla) S = 0,\\
&\partial_t \frac{\boldsymbol{B}}{\rho} + (\boldsymbol{V} \cdot \nabla)\frac{\boldsymbol{B}}{\rho} = \left(\frac{\boldsymbol{B}}{\rho} \cdot \nabla\right) \boldsymbol{V},\\
&P = P(\rho, S).
\end{aligned}\right\} \tag{4.29}$$

This system yields, by definition, the following three starting invariants:

$$I_0 = S(\boldsymbol{x}, t), \quad \boldsymbol{J}_0 = \boldsymbol{B}/\rho, \quad \rho_0 = \rho(\boldsymbol{x}, t). \tag{4.30}$$

Let us present a more detailed derivation of the starting $\boldsymbol{S}$-invariant. An equation for the vector potential $\boldsymbol{A}$ follows from equation for the magnetic field. It has the form

$$\partial_t A_i + (\boldsymbol{V} \cdot \nabla) A_i + A_m \frac{\partial V^m}{\partial x_i} = -\nabla(\phi - \boldsymbol{V} \cdot \boldsymbol{A}).$$

Here ϕ is an arbitrary function, chosen in accordance with the vector potential gauge. From the Coulomb gauge $\operatorname{div} \boldsymbol{A} = 0$, for example, it follows that $\Delta\phi = \operatorname{div}(\boldsymbol{V} \times \boldsymbol{B})$. One can, however, choose another one corresponding to $\phi = \boldsymbol{A} \cdot \boldsymbol{V}$. Then the equation for $\boldsymbol{A}$ transforms to

$$\partial_t A_i + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) A_i + A_m \frac{\partial V_m}{\partial x_i} = 0,$$

in which one can easily recognize an equation for the $\boldsymbol{S}$-invariant. Thus, the vector potential is the starting $\boldsymbol{S}$-invariant in the chosen gauge:

$$\boldsymbol{S}_0 = \boldsymbol{A}. \tag{4.31}$$

It is obviously possible to obtain this invariant in the usual Coulomb gauge of vector potential $\operatorname{div} \boldsymbol{A} = 0$. Then

$$\boldsymbol{S}_0 = \boldsymbol{A} - \boldsymbol{\nabla}\psi. \tag{4.32}$$

Here $\boldsymbol{A}$ is the vector potential in the Coulomb gauge and ψ is the function defined by

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} + \psi - \boldsymbol{V} \cdot \boldsymbol{A} = 0,$$

where $\mathrm{d}/\mathrm{d}t$ is the total derivative and $\Delta\phi = \operatorname{div}(\boldsymbol{V} \times \boldsymbol{A})$. Equations (4.30) and (4.31) can be chosen as a basic set of invariants. Then the new first-generation invariants have the form:

$$\boldsymbol{S}' = \boldsymbol{\nabla} S(\boldsymbol{x}, t), \quad I' = \frac{\boldsymbol{B} \cdot \boldsymbol{A}}{\rho}, \quad \rho' = \boldsymbol{B} \cdot \boldsymbol{A}. \tag{4.33}$$

The physical meaning of the invariant density is the helicity density of the magnetic field. In the second generation we have

$$I'' = \frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla} S, \quad \boldsymbol{S}'' = \boldsymbol{\nabla} \frac{\boldsymbol{B} \cdot \boldsymbol{A}}{\rho}, \quad \boldsymbol{J}'' = \frac{\boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{A}}, \quad \rho'' = \boldsymbol{B} \cdot \boldsymbol{\nabla} S, \quad \boldsymbol{J}_1 = \frac{1}{\rho} [\boldsymbol{\nabla} S \times \boldsymbol{A}]. \tag{4.34}$$

The geometrical nature of I'' is analogous to the Ertel invariant in compressible fluid. In the third generation, the invariants are much more abundant:

$$\left.\begin{aligned} I''' &= \frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla}\left(\frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} S}{\rho}\right), \quad \rho''' = \boldsymbol{B} \cdot \boldsymbol{\nabla}\left(\frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} S}{\rho}\right), \quad \boldsymbol{J}''' = \frac{1}{\rho}\left(\boldsymbol{\nabla} S \times \boldsymbol{\nabla}\left(\frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} S}{\rho}\right)\right), \\ \boldsymbol{S}''' &= \frac{1}{\rho}\left(\boldsymbol{B} \times \left(\boldsymbol{\nabla} S \times \boldsymbol{\nabla}\left(\frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} S}{\rho}\right)\right)\right), \end{aligned}\right\} \tag{4.35}$$

$$\left.\begin{aligned} \boldsymbol{S}_1''' &= \frac{1}{\rho}(\boldsymbol{B} \times (\boldsymbol{\nabla} S \times \boldsymbol{A})), \quad \boldsymbol{J}_1''' = \frac{\boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{A}} \times (\boldsymbol{\nabla} S \times \boldsymbol{A}), \quad \rho_1''' = \operatorname{div}\left(\frac{\rho \boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{A}}\right), \\ I_1''' &= \frac{\boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{A}} \cdot \boldsymbol{\nabla}\left(\frac{\boldsymbol{B} \cdot \boldsymbol{A}}{\rho}\right), \end{aligned}\right\} \tag{4.36}$$

$$I_2''' = \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} S}{\boldsymbol{B} \cdot \boldsymbol{A}}, \quad \boldsymbol{J}_2''' = \frac{1}{\rho}\left(\boldsymbol{A} \times \boldsymbol{\nabla} \frac{\boldsymbol{B} \cdot \boldsymbol{A}}{\rho}\right). \tag{4.37}$$

Here we have mentioned only some new invariants of ideal MHD. Multiplication of these invariants by an arbitrary function of Lagrangian invariants, for example, by $f(S, I, I'', \ldots)$ cannot violate its invariance. Further examples can be also easily constructed.

We have thus proved the following theorem:

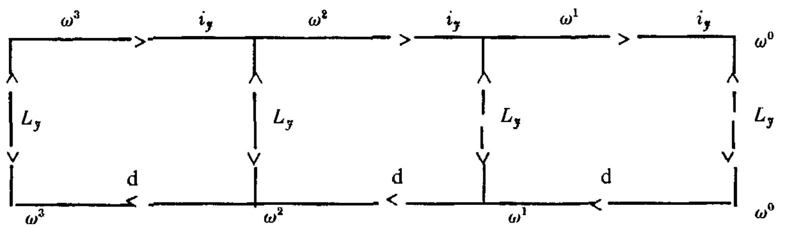
(i) The exterior product operation transforms invariant n-forms (that is, the forms satisfying equation $\partial_t \omega + L_V \omega = 0$) into invariant $(n+1)$-forms.

(ii) The interior product operation i_J (where J is a frozen-in field) transforms invariant n-forms into invariant $(n-1)$-forms.

(iii) The Lie differentiation operation along frozen-in fields transforms invariant n-forms into invariants n-forms.

(iv) An exterior product of invariant forms is an invariant form.

Transformation of invariant forms into invariant forms can be illustrated using the following diagram:



Here the points indicate invariant forms of the corresponding degree, arrows indicate transformation direction of the operation whose notation is indicated at the line.

5. Integral invariants

The formulation of dynamic invariants in terms of invariant forms has another important property. It enables one to evaluate integrals of invariant differential forms over regions of various dimensions, that is to introduce natural integral invariants. As a matter of fact, the theory of differential forms has developed just from the theory of manifold integration (Arnol'd 1978; von Westenholz 1981; Flanders 1989; Schutz 1982). The classical definition of the integral,

$$\int_{D_3} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3,$$

indicates that the usual integral over regions in R^3 is in fact an integral of a differential form. As is well known from calculus transformation of variables $x_i = x_i(y)$ yields the factor $J = \det\{\partial x_i / \partial y_k\}$, known as the Jacobian.

It is easy to notice that the same factor arises due to transformation variables of a basic differential form $dx_1 \wedge dx_2 \wedge dx_3$. So, the usual integral above mentioned is an integral of a 3-form, that is

$$\int_{D_3} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \equiv \int_{D_3} f(x_1, x_2, x_3)\, dx_1 \wedge dx_2 \wedge dx_3 = \int_{D_3} \omega^3.$$

We could say that the differential-forms language legalizes the mathematical nature of explicit integration. The integral of differential forms with $p < 3$ over regions in R^3, $\int_{D^p} \omega^p$ with D^p the p-dimensional surface or region, is defined in the same way (Arnol'd 1978; Flanders 1989; Schutz 1982).

The existence of invariant forms which obey (3.17) allows the introduction of integral conservation laws of the form

$$I^p = \int_{D^p(t)} \omega^p. \tag{5.1}$$

Here ω^p is an invariant p-form and $D^p(t)$ is the p-dimensional region advected by fluid motions.

Let us show that $I^p(t)$ is an invariant. Consider the quantity $I^p(t)$ in (5.1) at the moment $t = 0$:

$$I^p(0) = \int_{D^p(0)} \omega^p(0).$$

The region $D^p(0)$ changes its position under the advection in accordance with the transformation $x_i = x_i(\boldsymbol{x}_0, t)$. Therefore, this transformation can be considered as a variable transformation law in the integral. The value of this integral is independent of the choice of variables, so that

$$I^p(0) = I^p(t). \tag{5.2}$$

We have thus proved that the quantities $I^p(t)$ are integral conservation laws, that is

$$\frac{\mathrm{d}I^p}{\mathrm{d}t} = 0. \tag{5.3}$$

To illustrate these arguments in the coordinate form, we prove the conservation of $\int_{D^3} \omega^3$:

$$I^3(t) = \int_{D^3(t)} \rho(x, t)\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

We perform the transformation of variables in the integral, which is connected with the transformation to Lagrangian coordinates,

$$x_i = x_i(\boldsymbol{x}_0, t).$$

Then
$$I^3(t) = \int_{D^3(0)} \rho(\boldsymbol{x}_0, t) \det\left\{\frac{\partial x_i}{\partial x_k^0}\right\} \mathrm{d}x_1^0 \wedge \mathrm{d}x_2^0 \wedge \mathrm{d}x_3^0.$$

We transform the integrand using (3.28) for an invariant density:

$$I^3(t) = \int_{D^3(0)} \rho(\boldsymbol{x}_0, t) \det\left\{\frac{\partial x_i^0}{\partial x_k}\right\} \det\left\{\frac{\partial x_i}{\partial x_k^0}\right\} \mathrm{d}x_1^0 \wedge \mathrm{d}x_2^0 \wedge \mathrm{d}x_3^0.$$

Taking into account that $\det A \det B = \mathrm{Det}\, AB$ and $\partial x_i^0/\partial x^k$ is the inverse matrix to $\partial x_k/\partial x_i^0$ (that is $(\partial x_i^0/\partial x^k)(\partial x_k/\partial x_e^0) = \delta_{ie}$) we have eventually that

$$I^3(t) = \int_{D^3(0)} \rho_0(\boldsymbol{x}_0)\, \mathrm{d}x_1^0 \wedge \mathrm{d}x_2^0 \wedge \mathrm{d}x_3^0 \equiv I^3(0).$$

Thus, we have proved that an integral conservation law is associated with any invariant form. Moreover, we can say that there exist three types of integral invariants in hydrodynamic models, depending on the degree of the associated differential form. The first type of invariant is

$$I^1 = \int_{D^1(t)} \omega_S^1.$$

Here $D_1(t)$ is a closed contour frozen into fluid. The physical meaning of this invariant is the conservation of circulation of the field $\boldsymbol{S}(\boldsymbol{x}, t)$, which is an $\boldsymbol{S}$-invariant. The second one is

$$I^2 = \int_{D^2(t)} \omega_J^2.$$

Here $D^2(t)$ is a surface comoving with the fluid. The quantity I^2 describes conservation of the frozen-in field flux. The last type of the invariant takes the form

$$I^3 = \int_{D^3(t)} \omega_\rho^3.$$

The physical sense of this conservation law is obvious: the quantity associated with the invariant density inside a volume, which comoves with the fluid, is independent of time.

Let us give examples of different types of integral conservation laws in compressible fluid theory and MHD, using the local dynamic conservation laws from the previous section. For convenience we present them in a form usual for hydrodynamics. In compressible adiabatic fluid we have

$$\left.\begin{aligned} I_1^1 &= \oint_{\gamma(t)} \boldsymbol{V}\cdot d\boldsymbol{l}, \quad I_2^1 = \oint_{\gamma(t)} f(\boldsymbol{V}-\nabla H)\cdot d\boldsymbol{l}, \quad I_3^1 = \oint_{\gamma(t)} f\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}\right)\cdot d\boldsymbol{l}, \\ I_4^1 &= \oint_{\gamma(t)} f\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot\left(\nabla S\times\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}\right)\right)\cdot d\boldsymbol{l}, \quad I_5^1 = \oint_{\gamma(t)} f\frac{\operatorname{rot}\boldsymbol{V}}{\rho}(\nabla S\cdot(\boldsymbol{V}-\nabla H))\,d\boldsymbol{l}. \end{aligned}\right\} \tag{5.4}$$

Here $\gamma(t)$ is a closed contour moving with the fluid, S is the entropy, f is an arbitrary function on Lagrangian invariants:

$$f = f\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot(\boldsymbol{V}-\nabla H)}{\rho}; S; \frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}; \frac{\nabla S\times\nabla I_e}{\rho}\cdot(\boldsymbol{V}-\nabla H);\ldots\right).$$

Only a few invariants of the first type I' are given here (the number of these invariants is infinite and one can easily find new examples using equations from the previous section):

$$\left.\begin{aligned} I_1^2 &= \int_{S(t)} f\operatorname{rot}\boldsymbol{V}\cdot d\boldsymbol{S}', \quad I_2^2 = \int_{S(t)} f\nabla S\times\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}\right)\cdot d\boldsymbol{S}', \\ I_3^2 &= \int_{S(t)} \frac{f}{\rho}\nabla S\times(\boldsymbol{V}-\nabla H)\cdot d\boldsymbol{S}', \quad I_4^2 = \int_{S(t)} f\rho\operatorname{rot}\left(\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\times\frac{1}{\rho}\left(\nabla S\times\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}\right)\right)\right)\cdot d\boldsymbol{S}'. \end{aligned}\right\} \tag{5.5}$$

Here integration is carried out over a surface $S(t)$ moving together with the medium. Also,

$$\left.\begin{aligned} I_1^3 &= \int_{\Omega(t)} f\rho\,d\boldsymbol{X}, \quad I_2^3 = \int_{\Omega(t)} f\operatorname{rot}\boldsymbol{V}\cdot(\boldsymbol{V}-\nabla H)\,d\boldsymbol{X}, \quad I_3^3 = \int_{\Omega(t)} f\nabla S\cdot\operatorname{rot}\boldsymbol{V}\,d\boldsymbol{X}, \\ I_4^3 &= \int_{\Omega(t)} f(\boldsymbol{V}-\nabla H)\cdot\left(\nabla S\times\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}\cdot\nabla S}{\rho}\right)\right)d\boldsymbol{X}, \\ I_5^3 &= \int_{\Omega(t)} f(\nabla S\times(\boldsymbol{V}-\nabla H))\cdot\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot(\boldsymbol{V}-\nabla H)\right)d\boldsymbol{X}, \end{aligned}\right\} \tag{5.6}$$

where $\Omega(t)$ is a three-dimensional volume moving together with the medium.

The aforementioned examples include some known integral conservation laws for special choices of f. So at $f = 1$, I_1^3 becomes the mass conservation law, I_2^3 coincides with the helicity integral (Moffatt 1969) $\int \boldsymbol{V}\cdot\operatorname{rot}\boldsymbol{V}\,d\boldsymbol{x}$, I_4^3 with $f = \Psi(\operatorname{rot}\boldsymbol{V}\cdot\nabla S/\rho) - g'(S)$ coincides with the hypervorticity (Kuroda 1990).

Let us now give examples of integral invariants in ideal MHD:

$$I_1^1 = \oint_{\gamma(t)} \Phi \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}, \quad I_2^1 = \oint_{\gamma(t)} \Phi \nabla S \cdot \mathrm{d}\boldsymbol{l}, \quad I_3^1 = \oint_{\gamma(t)} \Phi \nabla \frac{\boldsymbol{B}\cdot\boldsymbol{A}}{\rho} \mathrm{d}\boldsymbol{l}, \tag{5.7}$$

$$\left.\begin{aligned} I_1^2 = \int_{S(t)} \Phi \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}', \quad I_2^2 = \int_{S(t)} \frac{\rho\Phi}{\boldsymbol{B}\cdot\boldsymbol{A}} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}', \\ I_3^2 = \int_{S(t)} \Phi[\nabla S \times \boldsymbol{A}]\cdot \mathrm{d}\boldsymbol{S}', \quad I_4^2 = \int_{S(t)} \Phi\left[\nabla S \times \left(\frac{\boldsymbol{B}\cdot\nabla S}{\rho}\right)\right]\cdot\mathrm{d}\boldsymbol{S}', \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} I_1^3 = \int_{\Omega(t)} \rho\Phi\,\mathrm{d}\boldsymbol{X}, \quad I_2^3 = \int_{\Omega(t)} \Phi \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}\boldsymbol{X}, \quad I_3^3 = \int_{\Omega(t)} \Phi \boldsymbol{B}\cdot\nabla\left(\frac{\boldsymbol{B}\cdot\nabla S}{\rho}\right)\mathrm{d}\boldsymbol{X}, \\ I_4^3 = \int_{\Omega(t)} \Phi \boldsymbol{A}\cdot\left(\nabla S \times \nabla\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\rho}\right)\right)\mathrm{d}\boldsymbol{X}. \end{aligned}\right\} \tag{5.9}$$

Here Φ is an arbitrary function of Lagrangian variables:

$$\Phi\left(\frac{\boldsymbol{B}\cdot\boldsymbol{A}}{\rho}; S; \frac{\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{A}}\cdot\nabla\frac{\boldsymbol{B}\cdot\boldsymbol{A}}{\rho}; \frac{\boldsymbol{B}\cdot\nabla S}{\rho}; \frac{\boldsymbol{B}}{\rho}\cdot\nabla\frac{\boldsymbol{B}\cdot\nabla S}{\rho}; \ldots\right)$$

The gauge of the vector potential $\boldsymbol{A}$ used in (5.7)–(5.9) is chosen to be the most convenient, as described in the previous section.

Equations (5.7) also contain known integral invariants when $\Phi = 1$, for example I_1^1 is the conservation of circulation of the vector potential, I_1^2 is the conservation of magnetic field flux, I_1^3 is the mass conservation law, and I_2^3 is the magnetic helicity conservation law (see e.g. Moffatt 1978). The number of examples can be easily increased with the use of equations from the previous section.

We have proved the existence of an infinite number of integral invariants of three types in hydrodynamic dissipationless media. It is important to notice that these integral invariants are important for stability analysis of different solutions with the help of the method proposed by Arnol'd (1969) and developed by Holm *et al.* (1985). In fact, these integrals can be included in the Liapunov functional with additional Lagrange multipliers when the Liapunov functional is to be found. These integrals contain arbitrary functions, which make them useful in stability analysis for a wide class of solutions. It is necessary to note that some of the integrals I^3 have a dynamic nature (e.g. I_1^3, the mass conservation law), and I_2^3 have a topological nature (e.g. I_2^3, (5.6), the Moffatt integral and I_2^3, (5.7), magnetic helicity conservations law).

6. Topological invariants

Consider now topological invariants in hydrodynamic models. Some examples of invariants of this type have been obtained in MHD (Woltjer 1958), incompressible fluid (Moffatt 1969) and two-fluid plasma (Sagdeev *et al.* 1986) models.

We start with a more simple question, that of the existence of charges of frozen-in fields. We have to use some well-known concepts.

The invariant p-form ω^p is called closed if $\mathrm{d}\omega^p = 0$. The forms satisfying the equation $\omega^p = \mathrm{d}\omega^{p-1}$ are referred to as exact. In accordance with the Poincaré Lemma (see e.g. Flanders 1989), in R^3 the form exists that satisfies $\omega^p = \mathrm{d}\omega^{p-1}$ for every closed form ω^p ($\mathrm{d}\omega^p = 0$). Thus, invariant forms of degree p are naturally divided into two classes, unclosed and closed. For invariant forms ω_S^1 this means that

S-invariants are of two types. Exact forms are associated with S-invariants of the gradient type (e.g. $\boldsymbol{S} = \nabla I$), and the unclosed ones (e.g. $\boldsymbol{S} = \boldsymbol{V} - \nabla H, \boldsymbol{S} = \boldsymbol{A}$) are associated with invariants which can be regarded as vector potentials of frozen-in integrals (since $\omega^2 = \mathrm{d}\omega^1_S$ is an invariant 2-form). Analogously, invariant two-forms are also divided into two types, namely the closed (exact) forms $\omega^2 = \mathrm{d}\omega^1$ possessing vector potentials (the latter being $\boldsymbol{S}$-invariants, that is, $\rho\boldsymbol{J} = \operatorname{rot}\boldsymbol{S}$, see (4.7)), and the unclosed invariant forms $\mathrm{d}\omega^2 \neq 0$. It is easy to see that all known frozen-in fields correspond to exact forms. For example, in compressible fluid, we have $\rho\boldsymbol{J} = \operatorname{rot}(\boldsymbol{V} - \nabla H)$, in ideal MHD $\rho\boldsymbol{J} = \boldsymbol{H}$, while for ideal two-fluid plasma hydrodynamics $\rho\boldsymbol{J}_\alpha = \operatorname{rot}(\boldsymbol{V}_\alpha - (e/mc)\boldsymbol{A})$ (see Sagdeev *et al.* 1986). In other words, all previously known frozen-in integrals are associated with exact forms $\omega^2 = \mathrm{d}\omega^1_S$. This leads to an important conclusion: such frozen-in fields do not have corresponding charges.

To verify this conclusion consider the flux of a frozen-in field through a closed surface ∂D^3 (we denote by ∂D the boundary of the region D) which encloses the volume D^3,

$$\int_{\partial D^3} \omega^2. \tag{6.1}$$

The frozen-in field $J\rho$ is chargeless when the flux through any closed surface equals zero. In terms of differential forms, Stokes' theorem acquires an especially elegant form:

$$\int_{\partial D} \omega = \int_D \mathrm{d}\omega \tag{6.2}$$

and embraces all the theorems on region integration known in classical calculus, such as Stokes', Gauss' and Green's theorems (Arnol'd 1978; Flanders 1989). According to (6.2), the integral (6.1) reduces to

$$\int_{\partial D^3} \omega^2 = \int_{D^3} \mathrm{d}\omega^2 = 0$$

since $\mathrm{d}\omega^2 = \mathrm{dd}\omega^1_S \equiv 0$ and ω^2 is closed. This proves that for a closed invariant 2-form the flux of a frozen-in field $J\rho$ vanishes, implying that these frozen-in fields are chargeless. In the coordinate form, the closed form of ω^2 implies that div $J\rho = 0$ which explicitly indicates the absence of charge. However, among the newly obtained frozen-in integrals, there are some associated with unclosed invariant 2-forms. These frozen-in fields possess charges. For example, the form (see (4.9))

$$\omega^2 = \omega^1_{S_1} \wedge \omega^1_{S_2},$$

is unclosed if $\omega^1_{S_1}$ is unclosed. In fact, $\mathrm{d}\omega^2 = \mathrm{d}\omega^1_{S_1} \wedge \omega^1_{S_2} - \omega^1_{S_1} \wedge \mathrm{d}\omega^1_{S_2} \neq 0$. Therefore the frozen-in fields (4.14) (see also (4.35)) are related to a new type of frozen-in charged field. This property evidently has a topological nature and leads to non-trivial integral invariants of the type

$$I^q = \int_{\partial D^3(t)} \omega^2 \neq 0, \tag{6.3}$$

which characterize the total charge within the region D^3. Some explicit examples of these invariants in compressible fluid are

$$I^q_1 = \oint_{s(t)} (\nabla S \times (\boldsymbol{V} - \nabla H) \cdot \mathrm{d}\boldsymbol{S}^1,$$

$$I^q_2 = \oint_{s(t)} (\boldsymbol{V} - \nabla H) \times \nabla \left[\frac{\operatorname{rot}\boldsymbol{V} \cdot \nabla S}{\rho}\right] \cdot \mathrm{d}\boldsymbol{S}^1.$$

In ideal MHD we obtain

$$I_1^q = \oint_{s(t)} (\boldsymbol{A} \times \nabla S) \cdot \mathrm{d}\boldsymbol{S}',$$

$$I_2^q = \oint_{s(t)} \left(\boldsymbol{A} \times \nabla \left(\frac{\boldsymbol{H} \cdot \nabla S}{\rho}\right)\right) \cdot \mathrm{d}\boldsymbol{S}'.$$

In these expressions, integration is carried out along a closed surface $s(t)$ moving with the fluid. Of course, only some simplest invariants of the new type are presented here; as previously, their number can be increased (see §5).

There arises a question: what are the topological invariants for frozen-in fields associated with closed 2-forms? The physical idea of introducing such invariants is rather simple. Since these frozen-in fields are chargeless, their field lines are either closed or extend to infinity. Therefore, when closed field lines are linked, then their frozenness prevents the flow from unlinking them, provided the flow is continuous. This is why the linkage of frozen-in field lines is also one of the topological invariants.

Particular examples of such invariants have been obtained in MHD (Woltjer 1958), for incompressible fluid (Moffatt 1969) and two-fluid plasmas (Sagdeev *et al.* 1986).

A detailed discussion of such invariants in incompressible media and their connection with topological principles can be found in Arnol'd (1974).

Following (Arnol'd 1974), we can define topological invariants for arbitrary hydrodynamic media (see Tur & Yanovsky 1991). Consider again a closed 2-form,

$$\omega_J = i_J \omega^3.$$

According to the Poincaré Lemma, a closed form on a smooth manifold is locally exact. Therefore, one can define a 1-form ω_*^1 such that $\omega_J^1 = \mathrm{d}\omega_*^1$. Because the Lie derivative and exterior differentiation d commute, one can always transform, choosing an appropriate gauge, the form ω_*^1 into an invariant 1-form frozen into the medium. Below we consider all ω_*^1 forms to be equivalent to an invariant 1-form ω_S^1.

Now define the 3-form and integral invariant as

$$I^\tau = \int_{D^3(t)} \omega_S^1 \wedge \mathrm{d}\omega_S^1. \tag{6.4}$$

Since ω_S^1 and $\mathrm{d}\omega_S^1$ are invariant forms, the 3-form $\omega_S^1 \wedge \mathrm{d}\omega_S^1$ is also invariant (see the theorem of §5). Hence, the quantity defined in (6.4) is an integral invariant,

$$\mathrm{d}I^\tau/\mathrm{d}t = 0. \tag{6.5}$$

When a natural metric exists in R^3, in terms of frozen-in fields $\boldsymbol{J}$, the invariant (6.4) takes the following coordinate form:

$$I^\tau = \int_{D^3(t)} \rho \boldsymbol{J} \cdot \mathrm{rot}^{-1}(\rho \boldsymbol{J}) \mathrm{d}\boldsymbol{X}. \tag{6.6}$$

We should stress that there are no additional restrictions on the region $D^3(t)$, such as the orthogonality of $\boldsymbol{J}$ and the normal to the boundary of $D^3(t)$. Conditions of this kind appear only in definitions of the Moffatt invariant and magnetic helicity. The absence of such conditions is connected with the fact that $\mathrm{rot}^{-1}(\rho \boldsymbol{J})$ is an $\boldsymbol{S}$-type invariant at a proper gauge choice.

The importance of the absence of restrictions on the surface of $D^3(t)$ becomes evident when the invariants are used for construction of the Liapunov functional (Holm *et al.* 1985).

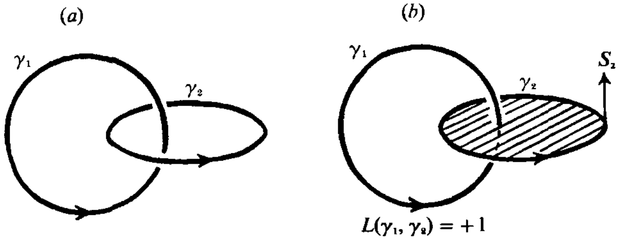


FIGURE 2. Illustration of the definition of the linking number for two contours γ_1 and γ_2, (a), as an algebraic summation over points at which the contour γ_1 intersets the surface associated with the contour γ_2; (b) S is a surface area vector whose direction is determined by the contour γ_2 orientation.

Let us now discuss the physical meaning of invariants (6.6) in terms of frozen-in fields. We consider two closed fields lines linked together (see figure 2a). The number of linkages $L(\gamma_1, \gamma_2)$ is defined in R^3 as (Flanders 1989; von Westenholz 1981)

$$L(\gamma_1,\gamma_2) = -\frac{1}{4\pi}\oint_{\gamma_1}\oint_{\gamma_2}\frac{(\boldsymbol{r}_2-\boldsymbol{r}_1)\cdot(\mathrm{d}\boldsymbol{r}_1\times \mathrm{d}\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}. \tag{6.7}$$

Here the integration is carried out along the lines γ_1 and γ_2, taking account of their natural orientations, as determined by the direction of the frozen-in field $\boldsymbol{J}$.

This linkage number can be equivalently redefined as follows. Consider a closed field line γ_2 and the surface with boundary γ_2; the field line direction is defined in accordance with the outer normal to the surface, $\boldsymbol{S}_2$. Moving along the contour γ_1 in a positive direction, one adds $+1$ when crossing the γ_2-surface along the $\boldsymbol{S}_2$ direction, and -1 when the crossing is in the opposite direction. The algebraic sum over all crossings of the surface by the contour γ_1 equals $L(\gamma_1, \gamma_2)$ (see figure 2b).

Consider now (following Moffatt 1969) the lines γ_1 and γ_2 as flux tubes of the frozen-in field $\boldsymbol{J}\rho$. Let the field $\boldsymbol{J}\rho$ be zero outside these tubes. Integration over the tube volume γ_2, for example, in the invariant (6.6) should be organized as an integration along the tube and then in the orthogonal direction:

$$\boldsymbol{J}\rho\,\mathrm{d}x^3 = \boldsymbol{J}\rho\,\mathrm{d}s\,\mathrm{d}\boldsymbol{r}_2.$$

Here $\mathrm{d}s$ is the cross-section of the flux tube γ_2, and the vector $\mathrm{d}\boldsymbol{r}_2$ is directed along the contour γ_2. Flux of the frozen-in field $\boldsymbol{J}\rho\,\mathrm{d}s$ is constant along the tube,

$$\boldsymbol{J}\rho\,\mathrm{d}x^3 = \Phi_2\,\mathrm{d}\boldsymbol{r}_2.$$

The integral (6.6), which now takes the form

$$I^{\tau} = \Phi_2\oint_{\gamma_2} \mathrm{rot}^{-1}(\boldsymbol{J}\rho)\cdot \mathrm{d}\boldsymbol{r}_2$$

can be transformed into a surface integral over the surface corresponding to the closed contour γ_2:

$$I^{\tau} = \Phi_2\oint_{\gamma_2} \boldsymbol{J}\rho\cdot \mathrm{d}\boldsymbol{s}.$$

From inspection of figure 2(b) it is easy to see that this integral can be expressed as

$$I^{\tau} = \Phi_1, \Phi_2 L(\gamma_1, \gamma_2).$$

These heuristic argument imply that the invariant I^{τ} is proportional to the linkage number for a frozen-in field. More rigorously, one can prove that I^{τ} is proportional

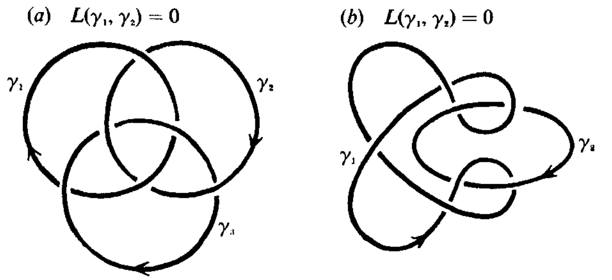


FIGURE 3. Oriented contour configurations with zero linking number: (*a*) the Borromean rings, linked in triplets rather than couples, (*b*), two contours linked together but with zero linking number.

to the average number of the field line linkages within the region D^3. Proof of this statement for an incompressible fluid is given by Arnol'd (1974) where the topological meaning of the invariant I^τ is treated in a more strict manner, together with its connection to the Hopf invariant (Bott & Tu 1982). In the more general case of an arbitrary hydrodynamic medium addressed here, these statements remain true.

Moreover, vanishing of I^τ does not mean topological triviality of the $J\rho$ field lines. In fact, a common example of non-trivially linked field lines with $I^\tau = 0$ is represented by the Borromean rings (see figure 3*a*). These rings are linked in triplets, rather than in couples. The reason for vanishing I^τ is obvious. It is also clear that only quadratic functions of the field $J\rho$ enter the invariant I^τ, so that this invariant is insensitive to triple linkages. However, the example of figure 3 shows that the integral I^τ can be equal to zero for non-trivial linkage of two lines. So, even for $I^\tau = 0$ the field line topology can be non-trivial.

It is important to notice that the set of frozen-in fields constructed in the previous section generates a large number of different topological invariants of the I^τ type. These topological invariants describe a number of topological constraints, and so give a more detailed description of non-trivial field topology in hydrodynamic media.

These topological invariants include the velocity field in combinations of order higher than 2, therefore these invariants are generally sensitive to triple, quadruple and higher linkages of field lines (e.g. Borromean ring and of higher-order configurations (Bott & Tu 1982; Berger 1990)).

Let us present some examples of invariants of this type. One can easily obtain the following integral for compressible fluid:

$$I_1^\tau = \int_{D^3(t)} \operatorname{rot} V\cdot(V-\nabla H)\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3.$$

This is a well-known Moffatt invariant (Moffatt 1969). Furthermore,

$$\left.\begin{aligned} I_2^\tau &= \int_{D^3(t)} \frac{1}{\rho}[\operatorname{rot} V\times(\nabla S\times(V-\nabla H))]\operatorname{rot}\frac{1}{\rho}[\operatorname{rot} V \\ &\qquad\qquad\times(\nabla S\times(V-\nabla H))]\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3, \\ I_3^\tau &= \int_{D^3(t)} \frac{1}{\rho}\left[\operatorname{rot} V\times(V-\nabla H)\times\frac{\operatorname{rot} V\cdot(V-\nabla H)}{\rho}\right]\operatorname{rot}\frac{1}{\rho}\left[\operatorname{rot} V\right. \\ &\qquad\left.\times\left((V-\nabla H)\times\nabla\frac{\operatorname{rot} V\cdot(V-\nabla H)}{\rho}\right)\right]\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3. \end{aligned}\right\} \tag{6.8}$$

The invariant I_2^τ describes an average number of line linkages of the frozen-in field

$$J' = \frac{1}{\rho}\operatorname{rot}\left[\frac{1}{\rho}\operatorname{rot} V \times (\nabla S \times (V - \nabla H))\right],$$

and I_3^τ is the same for the frozen-in field

$$J'' = \frac{1}{\rho}\operatorname{rot}\left[\frac{1}{\rho}\operatorname{rot} V \times \left[(V-\nabla H)\times \nabla\left[\frac{\operatorname{rot} V\cdot(V-\nabla H)}{\rho}\right]\right]\right].$$

In ideal MHD analogous integrals take the form

$$\left.\begin{aligned} I_1^\tau &= \int_{D^3(t)} \boldsymbol{H}\cdot\boldsymbol{A}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,\\ I_2^\tau &= \int_{D^3(t)} \frac{1}{\rho}[\boldsymbol{H}\times(\boldsymbol{A}\times\nabla S)]\operatorname{rot}\left[\frac{\boldsymbol{H}\times(\boldsymbol{A}\times\nabla S)}{\rho}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,\\ I_3^\tau &= \int_{D^3(t)} \frac{1}{\rho}\left[\boldsymbol{H}\times\left(\boldsymbol{A}\times\nabla\left(\frac{\boldsymbol{H}\cdot\boldsymbol{A}}{\rho}\right)\right)\right]\operatorname{rot}\left[\frac{\boldsymbol{H}\times\left(\boldsymbol{A}\times\nabla\left(\frac{\boldsymbol{H}\cdot\boldsymbol{A}}{\rho}\right)\right)}{\rho}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \end{aligned}\right\} \tag{6.9}$$

The number of examples of topological invariants in these media, of course, can be easily increased.

It should be noted that the nature of the invariants presented above is closely connected with the existence of one-dimensional foliation frozen into the media (the evolution of which reduces to their advection by fluid motions). The role of folia in these foliations is the same as that of the field (integral) lines of frozen-in fields. There exist certain topological prohibition rules associated with non-trivial configurations of these foliations conserving their type in the course of evolution (linkage of folia and knotedness of folia). This physical reason generates a large number of topological conservation laws associated with separate field lines, that is 'local' topological invariants. From this point of view, the problem of classification of one-dimensional foliations (Tamura 1979) as a whole acquires great importance as a source of global topological invariants.

Let us now discuss another, in our view more direct, interpretation of the invariant

$$I^\tau = \int_{D^3(t)} \omega_S^1 \wedge \mathrm{d}\omega_S^1$$

in terms of S-invariants. We can present it in the coordinate form with the use of S-invariants as

$$I^\tau = \int_{D^3(t)} \boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S}\,\mathrm{d}\boldsymbol{X}. \tag{6.10}$$

We have used here an expression for an invariant 1-form $\omega^1 = S_i\,\mathrm{d}x^i$. As noted in the §3, the $\boldsymbol{S}$-type invariants form a density field frozen into the medium. In terms of the 1-form, at a given invariant form ω_S^1, the equation for this field is

$$\omega_S^1 = S_i\,\mathrm{d}x^i = 0. \tag{6.11}$$

This is the Pfaff equation (Flanders 1989; Schutz 1982) which determines planes orthogonal to the field $\boldsymbol{S}(\boldsymbol{x},t)$ at each point. In contrast to vector fields, the existence of integral surfaces to the field of planes governed by (6.11) requires the Frobenius condition (Flanders 1989; Schutz 1982) to be satisfied. In terms of forms, we can present this condition as

$$\omega_S^1 \wedge \mathrm{d}\omega_S^1 = 0. \tag{6.12}$$

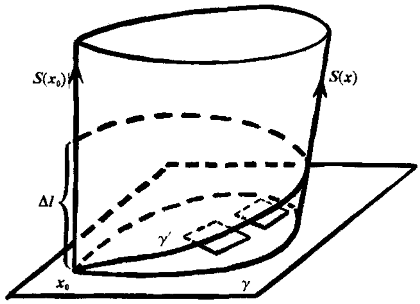


FIGURE 4. The non-integrability of the field of planes; Δl characterizes the non-integrability measure of the field of planes at the point x_0.

The field of planes is integrable so that there exists an integral surface tangent to the field of planes) provided only the Frobenius conditions (6.12) is satisfied. We can rewrite this condition in terms of S-type invariants as

$$\boldsymbol{S}\cdot \operatorname{rot} \boldsymbol{S} = 0. \tag{6.13}$$

One can understand the impediment to integration of the field of planes in the following purely geometric sense. Consider a field of planes in R^3. Choose a point $\boldsymbol{x}_0$ and consider the plane associated with the field $\boldsymbol{S}(\boldsymbol{x}_0)$ (see figure 4). Choose a closed contour γ passing through the point $\boldsymbol{x}_0$. The field $\boldsymbol{S}(\boldsymbol{x})$ is defined at each contour point, $\boldsymbol{x}\in\gamma$. Draw a line through each point $\boldsymbol{x}\in\gamma$ of the contour γ along the $\boldsymbol{S}(\boldsymbol{x})$ direction to obtain a cylindric surface (see figure 4). A section through this cylinder by the field of planes defines a field of directions on it. Consider now an integral line of this field of directions (which always exists), passing through the point $\boldsymbol{x}_0$ on this cylindric surface. Having made a turn around the cylinder, the integral line γ^1 in general does not reach the point $\boldsymbol{x}_0$ again. This is just the impediment mentioned above to the existence of integral surfaces to the field of planes. Indeed, if an integral surface existed, the line γ^1 would be situated on it and would naturally reach the point $\boldsymbol{x}_0$, when passing along the path γ^1. Thus, the local criterion of integrability for the field of plains can be naturally introduced in terms of the limiting value of the ratio of Δl (the distance between the starting and ending points of the integral line γ^1) and the surface area cut out by the contour γ when its radius tends to zero, that is

$$\lim_{\gamma\to x_0} \frac{\Delta l}{\Delta\sigma}. \tag{6.14}$$

Consider the case of a sufficiently small path γ and define circulation of the field $\boldsymbol{S}$ along the closed contour γ'' consisting of the curve γ^1 and the line connecting the starting and ending points of the path γ^1: $\int_{\gamma''}\boldsymbol{S}\,\mathrm{d}l$. Since the path γ^1 is tangent to the field of plains by definition, it is orthogonal to the field $\boldsymbol{S}$. Therefore, the sole contribution to this integral is due only to the line joining the outmost points of the contour γ^1. Then we have

$$\int_{\gamma'} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{l} = -|\boldsymbol{S}|\,\Delta l. \tag{6.15}$$

Here Δl is the distance between the outmost points of the contour γ^1 (see figure 4) tangent to the field $\boldsymbol{S}(\boldsymbol{x}_0)$.

On the other hand, using Stokes' theorem implies

$$\int_{\gamma''} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{l} = \int_{s(\gamma)} \operatorname{rot} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{S}'.$$

Taking into account the collinearity of $\mathrm{d}\boldsymbol{S}'$ and $\boldsymbol{S}$, we write $\mathrm{d}\boldsymbol{S}'$ as $\mathrm{d}\boldsymbol{S}' = \mathrm{d}\sigma/|\boldsymbol{S}|\,\boldsymbol{S}$. The integral then becomes

$$\int_{\gamma''} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{l} = \int_{s(\gamma)} \boldsymbol{S}\cdot \operatorname{rot}\boldsymbol{S}\frac{\mathrm{d}\sigma}{|\boldsymbol{S}|}.$$

If the circuit γ is sufficiently small, the integral can be evaluated as

$$\int_{\gamma} \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{l} = (\boldsymbol{S}\cdot \operatorname{rot}\boldsymbol{S})\frac{\Delta\sigma}{|\boldsymbol{S}|}. \tag{6.16}$$

Comparing (4.22) and (4.23) we obtain

$$\boldsymbol{S}\cdot \operatorname{rot}\boldsymbol{S} = -|\boldsymbol{S}|^2 \lim_{\gamma\to x_0}\frac{\Delta l}{\Delta\sigma}. \tag{6.17}$$

We see that this quantity characterizes the local criterion of non-integrability of the field of planes.

The form $\omega_S^1 \wedge \mathrm{d}\omega_S^1$ provides in invariant measure of the degree of non-integrability of the field of planes defined by the 1-form ω_S^1. In the coordinate form, we have $\rho_S = \boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S}$. So the invariant I^{τ} characterizes non-integrability of frozen-in field of planes associated with an $\boldsymbol{S}$-type invariant. This interpretation is convenient for the introduction of new topological invariants. Consider a situation when $I^{\tau} = 0$ due to vanishing of the density $\rho_S(\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0)$ in the whole region $D^3(t)$. It should be noted that the condition $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0$ is consistent with the medium dynamics, that is if in the region $D^3(0)$ (or R^3) we have $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0$ at $t = 0$, then $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0$ in the region $D^3(t)$ (or R^3) at any moment t. This is a consequence of invariance of the 3-form $\omega_S^1 \wedge \mathrm{d}\omega_S^1$ or, in other words, the statement is true because the density $\rho_S = \boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S}$ is governed by the continuity equation.

Making use of (2.14) one readily obtains the following equation for $\operatorname{rot}\boldsymbol{S}$:

$$\frac{\partial}{\partial t}\operatorname{rot}\boldsymbol{S} + (\boldsymbol{V}\cdot\nabla)\operatorname{rot}\boldsymbol{S} + \operatorname{rot}\boldsymbol{S}\operatorname{div}\boldsymbol{V} = (\operatorname{rot}\boldsymbol{S}\cdot\nabla)\,\boldsymbol{V}.$$

Differentiating $\rho_S \equiv \boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S}$ with respect to t and substituting the result into the time derivatives $(\partial/\partial t)\boldsymbol{S}$ and $(\partial/\partial t)\operatorname{rot}\boldsymbol{S}$ with the aid of (2.14) and the previous one, we obtain the continuity equation for ρ_S in the form

$$\frac{\partial}{\partial t}\rho_S + \operatorname{div}(\boldsymbol{V}\rho_S) = 0.$$

Thus, if the density ρ_S equals zero at $t = 0$, $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S}|_{t=0} = 0$, it remains zero for all $t > 0$: $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} \equiv 0$. So, if $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0$ and $I^{\tau} \equiv 0$ in a region $D^3(t)$ (or in all R^3), then the field of planes defined by an $\boldsymbol{S}$-type invariant is integrable, and the integral planes are frozen into the medium. This is why topological invariants appear, characterizing topological types of these two-dimensional surfaces, which, by virtue of the above arguments, cannot be modified by any fluid motions. It is easy to notice that these invariants are analogous to the local topological invariants characterizing the type of frozen-in field line (for example, a knot). Classification of two-dimensional surfaces is completely studied in topology, and such a topological invariant is well-known as the king of surface (Bott & Tu 1982). It can be also presented in an integral form using the Gauss–Bonne theorem (Flanders 1989; Schutz 1982).

We are interested, however, in another, more approximate invariant, characterizing a foliation into integral surfaces tangent to the frozen-in field of planes in region $D^3(t)$ (or in the whole R^3). It follows from the Frobenius condition that there exists a 1-form ω_η^1 such that

$$\mathrm{d}\omega_S^1 = \omega_\eta^1 \wedge \omega_S^1. \tag{6.18}$$

Let us apply the operator $\partial_t + L_V$ to the form $\mathrm{d}\omega_S^1$. Using invariance of both the ω_S^1-form and $\mathrm{d}\omega_S^1$-form (that is $\partial_t \omega_S^1 + L_V \omega_S^1 = 0$), we have

$$(\partial_t \omega_\eta^1 + L_V \omega_\eta^1) \wedge \omega_S = 0. \tag{6.19}$$

It follows from this equation that the 1-form ω_η^1 satisfies equations of the type

$$\partial_t \omega_\eta^1 + L_V \omega_\eta^1 = \alpha \omega_S^1. \tag{6.20}$$

Here α is an arbitrary function.

Let us define a 3-form with the use of ω_η^1:

$$\omega_\eta^3 = \omega_\eta^1 \wedge \mathrm{d}\omega_\eta^1. \tag{6.21}$$

One can easily prove, taking into account (6.20) and (6.18), that the form ω_η^3 is governed by the equation

$$\partial_t \omega_\eta^3 + L_V \omega_\eta^3 = -\mathrm{d}(\alpha \mathrm{d}\, \omega_S^1). \tag{6.22}$$

Let us now define an integral invariant

$$I^g = \int_D \omega_\eta^1 \wedge \mathrm{d}\omega_\eta^1. \tag{6.23}$$

Assuming that D is a three-dimensional compact manifold one can easily prove conservation of the quantity I^g. In fact, applying a differentiation operator with respect to t and using (6.22) one obtains

$$\frac{\partial I^g}{\partial t} = \int_D \partial_t \omega_\eta^3 = -\int_D \mathrm{d}(\alpha d\omega_S^1 + i_V \omega_\eta^3).$$

Using Stokes' theorem (6.2), the right-hand side can be transformed:

$$\frac{\partial I^g}{\partial t} = -\int_{\partial D} (\alpha d\omega_S^1 + i_V \omega_\eta^3).$$

Since for a compact manifold D its boundary ∂D equals zero, one obtains

$$\frac{\partial I^g}{\partial t} = 0.$$

So we have proved conservation of topological invariants I^g appearing in hydrodynamic media, when the Frobenius condition $\boldsymbol{S}\cdot \mathrm{rot}\, \boldsymbol{S} = 0$ is satisfied, and hence, when the corresponding topological invariant I^r vanishes. In this sense invariants I^g are complementary to the invariants I^r (6.4), and arise only when the latter tend to zero. In differential topology (Tamura 1979) this topological invariant I^g (6.23) is referred to as a Godbillon–Vey number. It characterizes the type of two-dimensional foliations defined by the form ω_S^1. One can also introduce integral invariants similar to (6.23), but with integration over local regions.

Recognizing the importance of the new topological invariant let us present a proof of its conservation in hydrodynamic media in a coordinate form. This proof demonstrates one more advantage of the use of forms in comparison with coordinate calculations. The proof and derivation of different equations are extremely simplified just in terms of forms. When there exist S-type fields (more exactly, invariant 1-forms), one can establish correspondence between the 1-form ω^1_η and the co-vector field η_i ($\omega^1_\eta = \eta_i\, dx^i$), if the metric exists. When $\boldsymbol{S}\cdot \operatorname{rot} \boldsymbol{S} = 0$ this correspondence is as follows:

$$\boldsymbol{\eta} = \frac{1}{\boldsymbol{S}^2}(\boldsymbol{S}\times \operatorname{rot}\boldsymbol{S}). \tag{6.24}$$

The invariant I^g, (6.23), is presentable in coordinates in the following form:

$$I^g = \int_{D^3(t)} \boldsymbol{\eta}\cdot \operatorname{rot}\boldsymbol{\eta}\, dx_1 \wedge dx_2 \wedge dx_3. \tag{6.25}$$

Here $D^3(t)$ is a three-dimensional region advected by the flow, defined in such way that the normal vector $\boldsymbol{n}$ to the closed surface surrounding the region $D^3(t)$ (that is $\partial D^3(t)$ is everywhere orthogonal to rot $\boldsymbol{S}$, that is $\boldsymbol{n}\cdot \operatorname{rot} \boldsymbol{S}|_{\partial D^3} = 0$. This condition means, that ∂D^3 coincides with one of the foliation foli formed by integral surfaces tangent to the field of planes ($\omega^1_S = 0$).

Let us derive an equation describing evolution of the field $\boldsymbol{\eta}$ (in terms of forms it corresponds to (6.20)) with the use of an equation defining an $\boldsymbol{S}$-type invariant (2.14). Introduce a convenient notations $\boldsymbol{\eta} = (1/S^2)(\boldsymbol{S}\times \operatorname{rot}\boldsymbol{S})$ and

$$\eta_i = \beta \boldsymbol{S}_k T_{ki}. \tag{6.26}$$

Here $\beta \equiv 1/\boldsymbol{S}^2$, $T_{ki} \equiv \partial S_k/\partial x_i - \partial S_i/\partial x_k$. Now using (2.14) it is easy to obtain an equation that describes the β and T_{ki} evolution. Differentiating η_i, (6.26), with respect to time and substituting derivatives $\partial\beta/\partial t$, $\partial s/\partial t$ and $\partial T_{ki}/\partial t$ using their evolution equations, it is easy to derive an equation for $\boldsymbol{\eta}$ in the form

$$\frac{\partial \eta_i}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\eta_i + \eta_m \frac{\partial V_m}{\partial x_i} = T_{ki}\left(S_k\, 2\beta^2 S_m S_e \frac{\partial V_e}{\partial x_m} - \beta S_m\left[\frac{\partial V_m}{\partial x_k} + \frac{\partial V_k}{\partial x_m}\right]\right). \tag{6.27}$$

Thus, it remains to prove that the terms on the right-hand side are transformed to the form αs. Using the identity

$$S_m T_{ki} \equiv T_{km} S_i + T_{k\alpha} S_\beta (\epsilon_{\alpha\beta p}\, \epsilon_{imp})$$

(here $\epsilon_{\alpha\beta\gamma}$ is the unit antisymmetric tensor) transform the terms on the right-hand side of (6.27) to obtain

$$\frac{\partial \eta_i}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\eta_i + \eta_m \frac{\partial V_m}{\partial x_i} = 2\beta^2 S_k S_e \frac{\partial V_e}{\partial x_m}(T_{km} S_i + T_{k\alpha} S_\beta\, \epsilon_{\alpha\beta p}\, \epsilon_{imp}) - T_{ki} S_m \left[\frac{\partial V_m}{\partial x_k} + \frac{\partial V_k}{\partial x_m}\right].$$

The terms have now acquired the necessary form. The latter terms should be presented as $2\beta(\boldsymbol{\eta}\times\boldsymbol{S})_p S_e (\partial V_e/\partial x_m)\, \epsilon_{imp}$.

Taking into account the constraint $\operatorname{rot}\boldsymbol{S} = \boldsymbol{\eta}\times\boldsymbol{S}$ (which follows from (6.18)) transform this equation to the form

$$\frac{\partial \eta_i}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\,\eta_i + \eta_m \frac{\partial V_m}{\partial x_i} = 2\beta^2 S_k S_e \frac{\partial V_e}{\partial x_m} T_{km} S_i + 2\beta(\operatorname{rot}\boldsymbol{S})_p\, \epsilon_{imp} S_e \frac{\partial V_e}{\partial x_m} - \beta T_{ki} S_m \left(\frac{\partial V_m}{\partial x_k} + \frac{\partial V_k}{\partial x_m}\right).$$

Using once again the identity

$$\epsilon_{ike}\epsilon_{pne} = \delta_{ip}\delta_{kn} - \delta_{in}\delta_{kp}$$

to transform the second term on the right-hand side, we obtain

$$\frac{\partial \eta_i}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\eta})\,\eta_i + \eta_m\frac{\partial V_m}{\partial x_i} = \left(2\beta S_e\frac{\partial V_e}{\partial x_m}\eta_m\right)S_i + \beta T_{mi}S_e\left(\frac{\partial V_e}{\partial x_m} - \frac{\partial V_m}{\partial x_e}\right).$$

Noticing that the last term on the right-hand side is equivalent to

$$\beta(\boldsymbol{S}\times\operatorname{rot}\boldsymbol{V})\times\operatorname{rot}\boldsymbol{S},$$

we obtain (with the use of $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} = 0$)

$$\left.\begin{aligned} &\partial_t\eta_i + (\boldsymbol{V}\cdot\nabla)\,\eta_i + \eta_m\frac{\partial V_m}{\partial x_i} = \alpha S_i,\\ &\alpha = \beta S_e\frac{\partial V_e}{\partial x_m}\eta_m - \beta(\operatorname{rot}\boldsymbol{V}\cdot\operatorname{rot}\boldsymbol{S}). \end{aligned}\right\} \qquad (6.28)$$

This equation coincides with (6.20).

To obtain equation for $\boldsymbol{p} = \operatorname{rot}\boldsymbol{\eta}$, apply the curl operation to (6.28):

$$\frac{\partial p_i}{\partial t} + (\boldsymbol{V}\cdot\nabla)\,p_i + \epsilon_{ike}\frac{\partial V_m}{\partial x_k}\frac{\partial \eta_e}{\partial x_m} + \epsilon_{ike}\frac{\partial \eta_m}{\partial x_k}\frac{\partial V_m}{\partial x_e} = \operatorname{rot}(\alpha\boldsymbol{S}).$$

Now use the identity

$$\frac{\partial V_m}{\partial x_k}\frac{\partial \eta_e}{\partial x_m} = \frac{\partial V_m}{\partial x_m}\frac{\partial \eta_e}{\partial x_k} + \epsilon_{\alpha\beta p}\,\epsilon_{kmp}\frac{\partial V_m}{\partial x_\alpha}\frac{\partial \eta_e}{\partial x_\beta}$$

to transform equation for $\boldsymbol{p}$ into a more convenient form:

$$\partial_t\boldsymbol{p} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{p} + \boldsymbol{p}\operatorname{div}\boldsymbol{V} - (\boldsymbol{p}\cdot\nabla)\,\boldsymbol{V} = \operatorname{rot}(\alpha\boldsymbol{S}). \qquad (6.29)$$

Multiplying (6.29) by η and (6.28) by $\boldsymbol{p}$ and summing the results, we obtain an equation describing the function $\psi = \boldsymbol{\eta}\operatorname{rot}\boldsymbol{\eta}$ which appears under the integration sign in invariant (6.25):

$$\partial_t\psi + \operatorname{div}(\psi\boldsymbol{V}) = \boldsymbol{\eta}\operatorname{rot}(\alpha\boldsymbol{S}).$$

It is easy to prove that $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{\eta} = 0$ (by means of evaluating the divergence of this equation and taking into account that $\operatorname{rot}\boldsymbol{S} = \boldsymbol{\eta}\times\boldsymbol{S}$). So the equation is presentable in a more convenient form as

$$\partial_t\psi + \operatorname{div}(\boldsymbol{V}\psi) = \operatorname{div}(\alpha\boldsymbol{S}\times\boldsymbol{\eta}). \qquad (6.30)$$

One can appreciate advantages of the language of forms when comparing the derivation of (6.30) in the coordinate form to the derivation of the same equation, (6.22), in terms of forms.

Differentiating I^g (6.25) with respect to time we have

$$\frac{\mathrm{d}I_g}{\mathrm{d}t} = \int_{D^3(t)} (\partial_t\psi + \operatorname{div}(\boldsymbol{V}\psi))\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3.$$

Taking into account (6.30) one can present the integrand as

$$\frac{\mathrm{d}I_g}{\mathrm{d}t} = \int_{D^3(t)} \operatorname{div}(\alpha\boldsymbol{S}\times\boldsymbol{\eta})\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3.$$

Now with the help of Stokes' theorem transform the volume integral into a surface one:

$$\frac{\mathrm{d}I_g}{\mathrm{d}t} = \int_{\partial D^3(t)} (\alpha \boldsymbol{S} \times \boldsymbol{\eta})\, \boldsymbol{n} \mathrm{d}S',$$

where $\boldsymbol{n}$ is the unit vector normal to the surface $\partial D^3(t)$, $\mathrm{d}S'$ is the element of surface area $\partial D^3(t)$. Since vector $\boldsymbol{n}$ is parallel to vector $\boldsymbol{S}$, the integral vanishes, implying

$$\frac{\mathrm{d}I_g}{\mathrm{d}t} = 0.$$

This accomplishes the proof of the fact that the invariant I^g, (6.25), is conserved for all hydrodynamic media. In geometrical terms, this invariant characterizes the degree of non-integrability of the field of planes $\omega^1_{\boldsymbol{\eta}}$ orthogonal at each point to an integrable field of planes defined by means of $\omega^1_{\boldsymbol{S}}$, or the $\boldsymbol{S}$-type invariant.

We have thus rigorously proved conservation of the invariant I^g. It only remains to realize that it does not vanish identically. For this purpose, it is sufficient to construct an example of the field with $I^g \neq 0$. Let us give such an example using a field $\boldsymbol{S}$ of the type $\boldsymbol{S} = \phi \nabla \psi$. This field satisfies the Frobenius equation $\boldsymbol{S} \cdot \operatorname{rot} \boldsymbol{S} = \phi \nabla \psi \cdot [\nabla \phi \times \nabla \psi] \equiv 0$, so there exists an integral surface for the field of planes orthogonal at each point to the vector field $\boldsymbol{S}$. In our example these are the surfaces where the function ψ is constant.

Let us choose for these functions the following ones: $\phi = x + z^4$ and $\psi = y + 1/z$. Evaluating the corresponding field $\boldsymbol{\eta} = (1/S^2)[\boldsymbol{S} \times \operatorname{rot} \boldsymbol{S}]$ according to its definition, it is easy to find the integrand of the invariant as

$$\boldsymbol{\eta} \cdot \operatorname{rot} \boldsymbol{\eta} = -\frac{4z^4(5+z^4)}{(x+z^4)^2 (1+z^4)^2}.$$

Since the quantity $\boldsymbol{\eta}$ rot $\boldsymbol{\eta}$ is strictly negative throughout the region $\boldsymbol{x} \in R^3$, it is clear that its integral cannot be zero. So we can say that there exist configurations of field $\boldsymbol{S}$ such that $\boldsymbol{S} \cdot \operatorname{rot} \boldsymbol{S} = 0$ (and correspondingly, $I^r \equiv 0$) and the value of the topological invariant I^g on them does not vanish.

With this we complete a rigorous proof that either invariant I^g is conserved or it is non-trivial.

Let us give the simplest examples of new topological invariants. If the equation $(\boldsymbol{V} - \nabla H) \operatorname{rot} \boldsymbol{V} = 0$ holds for compressible adiabatic fluid in a region $D^3(t)$ (or R^3), and hence the Moffatt invariant vanishes,

$$I^r = \int_{D^3} (\boldsymbol{V} - \nabla H) \cdot \operatorname{rot} \boldsymbol{V} \mathrm{d}x \equiv 0$$

a new topological invariant appears:

$$I^g_0 = \int_{D^3} \frac{(\boldsymbol{V} - \nabla H) \times \operatorname{rot} \boldsymbol{V}}{(\boldsymbol{V} - \nabla H)^2} \cdot \operatorname{rot} \left(\frac{(\boldsymbol{V} - \nabla H) \times \operatorname{rot} \boldsymbol{V}}{(\boldsymbol{V} - \nabla H)^2} \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \qquad (6.31)$$

where vector $\boldsymbol{n}$ normal to the surface $\partial D^3(t)$ is everywhere orthogonal to rot $\boldsymbol{V}$,

$$\boldsymbol{n} \cdot \operatorname{rot} \boldsymbol{V}|_{\partial D^3(t)} = 0.$$

In this model there exist $\boldsymbol{S}$-type invariants for which the Frobenius condition is satisfied automatically. For example $\boldsymbol{S}_1 = \boldsymbol{I} \nabla S$ and $\boldsymbol{S}_2 = S \nabla \tilde{I}$, where

$$\tilde{I} = (\boldsymbol{V} - \nabla H) \cdot \operatorname{rot} \boldsymbol{V} / \rho.$$

For such $\boldsymbol{S}$-type invariants the topological invariants vanish, $I^{\tau} \equiv 0$. That is why only topological invariants I^{g} are non-trivial for them. For the above-mentioned fields the I^{g} invariants are as follows:

$$I_1^g = \int_{D^3(t)} \frac{\nabla\tilde{I}\cdot\left(\nabla S\times\nabla\left(\frac{\nabla S\cdot\nabla\tilde{I}}{(\nabla S)^2}\right)\right)}{\tilde{I}^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{6.32}$$

$$I_2^g = \int_{D^3(t)} \frac{\nabla S\cdot\left(\nabla\tilde{I}\times\nabla\left(\frac{\nabla\tilde{I}\cdot\nabla S}{(\nabla\tilde{I})^2}\right)\right)}{S^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3. \tag{6.33}$$

In the invariants $I^{g}{}_{1,2}$, vector $\boldsymbol{n}$ normal to $\partial D^3(t)$ is orthogonal to $(\nabla\tilde{I}\times\nabla S)$, that is

$$\boldsymbol{n}\cdot(\nabla\tilde{I}\times\nabla S)\,|_{\partial D^3(t)} = 0.$$

A new topological invariant also appears in ideal MHD with $\boldsymbol{A}\cdot\boldsymbol{H} = 0$ and $I^{\tau} = \int \boldsymbol{A}\cdot\boldsymbol{H}\,\mathrm{d}\boldsymbol{x} \equiv 0$ (here, as above $\boldsymbol{A}$ is presented in the most suitable gauge):

$$I^g = \int_{D^3(t)} \frac{\boldsymbol{A}\times\boldsymbol{H}}{A^2}\cdot\operatorname{rot}\left(\frac{\boldsymbol{A}\times\boldsymbol{H}}{A^2}\right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{6.34}$$

where vector $\boldsymbol{n}$ normal to the surface $\partial D^3(t)$ is also orthogonal to $\boldsymbol{H}$.

In MHD there also exist ρ-type invariants for which $\boldsymbol{S}\cdot\operatorname{rot}\boldsymbol{S} \equiv 0$. For these fields the invariants I^{τ} are trivial, $I^{\tau} \equiv 0$, and so there exist only topological invariants I^{g}. As examples of such fields in MHD, one can consider

$$\boldsymbol{S}' = S\nabla\left(\frac{\boldsymbol{A}\cdot\boldsymbol{H}}{\rho}\right),\quad \boldsymbol{S}'' = \left(\frac{\boldsymbol{A}\cdot\boldsymbol{H}}{\rho}\right)\nabla S,\quad \boldsymbol{S}''' = \left(\frac{\boldsymbol{A}\cdot\boldsymbol{H}}{\rho}\right)\nabla\left(\frac{\boldsymbol{H}}{\rho}\cdot\nabla\left(A\frac{\boldsymbol{H}}{\rho}\right)\right).$$

For these fields the topological invariants I^{g} are

$$I^{g'} = \int_{D^3(t)} \frac{\nabla S\cdot\left(\nabla I'\times\left(\frac{\nabla I'\cdot\nabla S}{(\nabla I')^2}\right)\right)}{S^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{6.35}$$

$$I^{g''} = \int_{D^3(t)} \frac{\nabla I'\cdot\left(\nabla S\times\nabla\cdot\left(\frac{\nabla S\cdot\nabla I'}{(\nabla I')^2}\right)\right)}{(I')^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{6.36}$$

$$I^{g'''} = \int_{D^3(t)} \frac{\nabla I'\cdot\left[\nabla\cdot\left(\frac{\boldsymbol{H}}{\rho}\nabla I'\right)\times\nabla\cdot\left(\frac{\nabla\left(\frac{\boldsymbol{H}}{\rho}\cdot\nabla I'\right)\cdot\nabla I'}{\left(\nabla\left(\frac{\boldsymbol{H}}{\rho}\cdot\nabla I'\right)\right)^2}\right)\right]}{(I')^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3, \tag{6.37}$$

where $I^{1} \equiv \boldsymbol{A}\cdot\boldsymbol{H}/\rho$ and the normal vector $\boldsymbol{n}$ to $\partial D^3(t)$ is orthogonal to the corresponding $\operatorname{rot}\boldsymbol{S}$.

Of course, we have presented only the simplest examples of new topological invariants. Their number can be easily enlarged with the use of the $\boldsymbol{S}$-type invariants and frozen-in fields obtained in the previous sections, while the theorem on invariant forms enables us to construct limitless number of hydrodynamic media topological invariants.

7. Results and discussion

Let us summarize and discuss the main results of the present paper.

1. All universal geometric relations between invariants are obtained in the §§3 and 4 based on the description in terms of invariant forms. Universality of these relations makes them applicable to any hydrodynamic dissipationless medium. It is possible, of course, to verify the relations obtained, avoiding the differential forms language, by their direct substitution into the equations that define the invariants in coordinate representation (2.8), (2.11), (2.14), (2.15). However, only with the aid of this language did it become clear that *all* possible relations between the invariants have been obtained.

Any other relations should coincide either with the ones obtained or with the result of their subsequent application. This is because there are no other operations on the form space apart from the previously listed ones.

2. A simple method of constructing new local dynamic invariants in hydrodynamic media from a limited basic set of known invariants is proposed.

In principle, one can use Lagrange variables $\boldsymbol{x}_0$ expressed in terms of Eulerian coordinates, $\boldsymbol{x}_0 = \boldsymbol{x}_0(\boldsymbol{x}, t)$, as a universal set of basic invariants. These expressions are, by definition, Lagrangian invariants for all hydrodynamic media. The only difficulty in this case consists of expressing the invariants obtained in terms of the basic Eulerian fields. Another possibility to choose the basic set is associated with the Lagrange or Hamilton formulations for hydrodynamic systems, for example, in terms of Klebsh variables (see also Holm *et al.* 1983).

It is interesting to note that the proposed method clarifies the common nature of invariants and relations between them that appear in different hydrodynamic media (e.g. between $I = (\operatorname{rot} \boldsymbol{V}/\rho)\cdot\nabla S$ and $I = (\boldsymbol{H}/\rho)\cdot\nabla S$). Moreover, it permits one to avoid the technically cumbersome verification of invariance when substituting them into the basic hydrodynamic equations.

3. With the use of this method a number of examples of new invariants in compressible adiabatic fluid and ideal MHD have been constructed.

4. In §4 it is shown that frozen-in integrals form a Lie algebra with the multiplication operation defined by a vector field commutator. This important property not only permits us to construct new frozen-in fields, but also generates a specific Backlund transformation for solutions of hydrodynamic systems of equations.

5. The language of differential forms makes it possible to formulate and obtain in the most natural manner integral invariants for hydrodynamic media. The conservation proofs for these invariants are extremely simple and compact in this language.

6. The subdivision of differential forms into closed and unclosed ones is of great importance for our study, because it leads to the existence of different types of frozen-in integrals and $\boldsymbol{S}$-type invariants. As shown in this paper, closed ω_J^2 forms are responsible for chargeless frozen-in fields, while unclosed ω_J^2-forms determine charged ones. For the latter frozen-in fields an integral invariant appears, the charge conservation law. So far there have been no examples of such invariants in hydrodynamics. We present here explicit examples of new charged frozen-in integrals in compressible fluid and MHD.

The mathematical nature of the subdivision of forms into closed, unclosed and exact ones is more profound than as discussed in the present paper, since it demands the introduction of somewhat more complicated topological concepts. On the basis of this subdivision, there is a close connection between differential forms and differential topology (Bott & Tu 1982). One can take as an example of such a connection the definition of a k-dimensional group of cohomologies of a manifold $\boldsymbol{M}$ which is one of the most important elements of topology. All the p-forms on a manifold form a linear space, while the closed differential forms form its subspace. The exact p-forms form a subspace of closed forms. A p-dimensional cohomology group of $\boldsymbol{M} H^p(\boldsymbol{M}, \boldsymbol{R})$ manifolds is called a group, elements of which are the equivalence classes of closed ω^p-forms differing in exact p-forms, that is $\{\omega_1^p\} \equiv \{\omega_1^p + \mathrm{d}\omega^{p-1}\}$. The dimension of $H^k(\boldsymbol{M}, \boldsymbol{R})$ is referred to as the k-dimensional Betty number; it is a topological invariant of $\boldsymbol{M}$.

7. Topological invariants conserved in all hydrodynamic media are constructed in this paper with the aid of invariant differential forms and their geometric meaning is discussed. There exist topological invariants I^r for closed ω_J^2-forms, that is for charged frozen-in fields associated with the conservation of the corresponding frozen-in field lines linkage number. The I^r invariant terms of $\boldsymbol{S}$-type invariants have another interpretation which is discussed in the paper. It is non-trivial only for unclosed ω_S^1 invariant 1-forms ($\mathrm{d}\omega_S^1 \neq 0$), i.e. for the $\boldsymbol{S}$-type invariants that can be vector potentials of frozen-in fields. I^r invariance characterizes the degree of non-integrability of the field of planes normal to the vector field of the $\boldsymbol{S}$-type invariant. If this field of planes is integrable, then $I^r \equiv 0$ and a new type of topological invariants, I^g, obtained in the present paper arise in hydrodynamic media. The invariant I^g characterizes a topological foliation type of integral planes of this field of planes. In this sense I^g is a complementary topological invariant to the I^r-invariant which is sensible to topological distinctions among the states with $I^r \equiv 0$.

If a form ω_S^1 is exact ($\omega_S^1 = \mathrm{d}f$), it does not form a non-trivial frozen-in field and the invariant I^r vanishes identically for all $\boldsymbol{S}$-type invariants configurations. For such $\boldsymbol{S}$-type invariants, there exist only new topological invariants I^g, i.e. the Godbillon–Vey numbers.

8. Recognizing the importance of a new topological invariant I^g we present a rigorous proof of its conservation and non-triviality in the coordinate representation. Comparison of the latter with the proof in terms of differential forms clearly demonstrates the simplicity and all the other advantages of the language of differential forms.

9. Explicit examples of new topological invariants of the I^r and I^g type in different hydrodynamic media are obtained in the paper. These examples are presented in the common coordinate form. The list of these invariants is limitless, since taking into account the results of §4, one can proceed in constructing more and more new $\boldsymbol{S}$-invariants and frozen-in integrals based on the one already constructed. Some explicit constructions are presented in the appendices for compressible and incompressible fluids.

Appendix A. Invariants in incompressible fluids

To illustrate the application of the general relations, we give here the invariants for incompressible fluids. The equation of motion can be written in terms of the velocity 1-form, $\omega_V^1 = V_i \,\mathrm{d}x^i$ as follows:

$$\partial_t \omega_V^1 + L_V \omega_V^1 = -\mathrm{d}\tilde{P}, \tag{A 1}$$

where $\tilde{P} = \frac{1}{2} V^2 + P$ is the renormalized pressure. The invariant 3-form of mass reduces in this case to the 3-form of volume and results in solenoidal velocity fields. The invariant 1-form ω_S^1 can be easily obtained after definition of the 0-form through

$$\partial_t H + L_V H = -\tilde{P}, \qquad \text{(A 2)}$$

where $H = \int \tilde{P}\,\mathrm{d}t$ is the so-called action with the Lagrangian $\tilde{P} = \frac{1}{2}V^2 + P$ defined by the Bernoulli integral. A similar action was used by Hollman (1964). On the other hand, one can consider H as a gauge function. In this gauge the velocity field is an S-invariant. Then

$$\omega_S^1 = \omega_V^1 - \mathrm{d}H. \qquad \text{(A 3)}$$

Evaluation of the external differential of (A 1) taking account of its commutation properties with L_V and $\mathrm{d}d = 0$ leads to the following invariant 2-form:

$$\omega^2 = \mathrm{d}\omega_V^1. \qquad \text{(A 4)}$$

In terms of the coordinates on R^3, the following frozen-in field can be obtained from (A 4):

$$\boldsymbol{J} = \mathrm{rot}\, \boldsymbol{V}.$$

With this form and with ω_S^1 we determine the Lagrangian invariant

$$I = i_J \omega_S^1. \qquad \text{(A 5)}$$

In the coordinate representation we have

$$\boldsymbol{J} = \mathrm{rot}\,\boldsymbol{V}; \quad \boldsymbol{S} = \boldsymbol{V} - \nabla H; \quad I = \mathrm{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V} - \nabla H). \qquad \text{(A 6)}$$

The list (A 6) can be easily extended with help of relations between invariants. Indeed (see §4),

$$\boldsymbol{S} = \nabla[\mathrm{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V} - \nabla H)], \qquad \text{(A 7)}$$

$$I' = \mathrm{rot}\,\boldsymbol{V}\cdot\nabla[\mathrm{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V} - \nabla H)], \qquad \text{(A 8)}$$

The invariants (A 7), (A 8) and similar ones contain higher derivatives of the fields when the equation of motion has the form (A 1), and they can be naturally called higher invariants. The number of such invariants can be further extended with use of relations obtained above (see the diagram at the end of §4).

Let us give some examples of S-type invariants. Start with new integral invariants that describe charge conservation laws of the corresponding frozen-in fields:

$$I_1^q = \oint_{S(t)} (\boldsymbol{V} - \nabla H) \times \nabla((\boldsymbol{V} - \nabla H)\cdot \mathrm{rot}\,\boldsymbol{V})\,\mathrm{d}\boldsymbol{S}'. \qquad \text{(A 9)}$$

This is the charge conservation law of the field $\boldsymbol{J}' = (\boldsymbol{V} - \nabla H)\times\nabla((\boldsymbol{V} - \nabla H)\cdot\mathrm{rot}\,\boldsymbol{V})$. For the frozen-in field $\boldsymbol{J}'' = (\boldsymbol{V} - \nabla H)\times\nabla I'$ (here is defined by (A 8)) the charge conservation law takes the form

$$I_2^q = \oint_{S(t)} (\boldsymbol{V} - \nabla H) \times \nabla(\mathrm{rot}\,\boldsymbol{V}\cdot\nabla(\mathrm{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V} - \nabla H)))\,\mathrm{d}\boldsymbol{S}'. \qquad \text{(A 10)}$$

Similarly, with the use of (5.1) one can easily construct other types of integral conservation laws: 'Mass' conservation laws, for example, that are described by the corresponding density invariants:

$$I^3 = \int_{V(t)} f(\boldsymbol{V}\cdot\mathrm{rot}\,\boldsymbol{V}, \mathrm{rot}\,\boldsymbol{V}\cdot\nabla(\mathrm{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V} - \nabla H)), \ldots)(\boldsymbol{V} - \nabla H)\cdot\mathrm{rot}\,\boldsymbol{V}\,\mathrm{d}X^3. \qquad \text{(A 11)}$$

Here f is an arbitrary function of Lagrangian invariants.

For an example of a topological invariant of I^r-type in this medium one can take (see (6.4))

$$I_1^r = \int_{V(t)} (\boldsymbol{V}-\nabla H)\cdot \text{rot}\, \boldsymbol{V} \,\mathrm{d}\boldsymbol{x}^3. \tag{A 12}$$

Then the integral (A 12) coincides with Moffatt's invariant if rot $\boldsymbol{V}$ is tangent to the surface bounding the volume $V(t)$.

When $(\boldsymbol{V}-\nabla H)\cdot\text{rot}\,\boldsymbol{V} = 0$, a new invariant arises:

$$I_1^2 = \int_{V(t)} \frac{[(\boldsymbol{V}-\nabla H)\times \text{rot}\,\boldsymbol{V}]}{(\boldsymbol{V}-\nabla H)^2}\cdot \text{rot}\left[\frac{(\boldsymbol{V}-\nabla H)\times \text{rot}\,\boldsymbol{V}}{(\boldsymbol{V}-\nabla H)^2}\right]\mathrm{d}\boldsymbol{x}^3. \tag{A 13}$$

The topological invariant (A 12) is complementary to (A 13).

The topological invariant I^r vanishes identically for a field $\boldsymbol{S}$ of the type $I\nabla I'$. For example,

$$\boldsymbol{S} = [\text{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V}-\nabla H)]\,\nabla[\text{rot}\,\boldsymbol{V}\cdot\nabla(\text{rot}\,\boldsymbol{V}\cdot(\boldsymbol{V}-\nabla H))]$$

(see (6.32) and (6.33)). Thus, for such S-fields only I^g-invariants are non-trivial.

Of course, we have presented only the simplest examples of invariants for incompressible fluid. It is easy to extend the list of these invariants with the aid of the results obtained in §4.

Appendix B. Compressible fluid

Equations of motion of compressible fluid can be represented in the form

$$\partial_t \omega_V + L_V \omega_V = -dp; \quad \partial_t \omega_\rho^3 + L_V \omega_\rho^3 = 0.$$

Here ρ follows from the equation of state, $p = p(\rho)$. An invariant 1-form can be easily obtained based on the gauge

$$\omega_S = \omega_V - \mathrm{d}H,$$

where the zero-form H is governed by the equation

$$\partial_t H + L_V H = -\tilde{P}.$$

The latter equation can be easily integrated in Lagrangian variables:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\tilde{P}; \quad H = -\int_0^t \tilde{P}(\tau)\,\mathrm{d}\tau.$$

We have once again obtained the action with the Lagrangian $\tilde{P} = \frac{1}{2}V^2 + p(\rho)$, which coincides with the Bernoulli integral. Thus, in the chosen gauge the velocity field is an $\boldsymbol{S}$-type invariant.

The frozen-in vector field can be obtained through the exterior differential of the equation of motion. Then comparison of the invariant 2-form $\omega^2 = \mathrm{d}\omega_V$ with the 2-form $i_J \omega_\rho^3$ leads to the well known result,

$$\boldsymbol{J} = \text{rot}\,\boldsymbol{V}/\rho.$$

Thus, we have obtained an invariant 1-form and a frozen-in vector field. As follows from the diagram in §4, there are two ways to obtain new invariants from $\omega_S = \omega_V - \mathrm{d}H$. The first route yields the following invariant forms:

$$\omega^0 = i_J\omega_S, \quad {}^{(1)}\omega^0 = L_J i_J \omega_S \equiv i_J \mathrm{d} i_J \omega_S, \quad {}^{(1)}\omega_S = \mathrm{d} i_J \mathrm{d} i_J \omega_S, \quad {}^{(2)}\omega_S = \mathrm{d} i_J \mathrm{d} i_J \mathrm{d} i_J \omega_S.$$

Thus, we obtain two Lagrangian invariants and two invariants 1-forms, ${}^{(1)}\omega_S$ and ${}^{(2)}\omega_S$. The second route gives

$$
{}^{(1)}\omega_S = L_J\omega_S, \quad {}^{(1)}\omega_S^2 = \mathrm{d}L_J\omega_S = \mathrm{d}i_J\mathrm{d}\omega_S,
$$
$$
{}^{(2)}\omega_S^2 = L_J\mathrm{d}L_J\omega_S = \mathrm{d}i_J\mathrm{d}i_J\mathrm{d}\omega_S, \quad {}^{(2)}\omega^1 = i_J L_J \mathrm{d}L_J\omega_S = i_J\mathrm{d}i_J\mathrm{d}i_J\mathrm{d}\omega_S.
$$

Two new invariant 2-forms determine new frozen-in field $\boldsymbol{J}'$ and $\boldsymbol{J}''$ as

$$
i_{J'}\omega_\rho^3 = {}^{(1)}\omega^2; \quad i_{J''}\omega_\rho^3 = {}^{(2)}\omega^2.
$$

Furthermore, as we have mentioned above, the frozen-in fields can be obtained from invariant 1-forms since the exterior product of two invariant 1-forms is an invariant 2-form. Basing on these frozen-in fields and invariant forms, one can obtain new invariants using the diagram.

The invariants obtained can be represented in the coordinate form as

$$
I^0 = (\boldsymbol{V}-\nabla H)\cdot\frac{\operatorname{rot}\boldsymbol{V}}{\rho}, \quad I' = \frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot\nabla(\boldsymbol{V}-\nabla H)\cdot\frac{\operatorname{rot}\boldsymbol{V}}{\rho},
$$
$$
\boldsymbol{S}^0 = \boldsymbol{V}-\nabla H, \quad \boldsymbol{S}^1 = \nabla\left[\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot\nabla\left((\boldsymbol{V}-\nabla H)\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\right)\right],
$$
$$
\boldsymbol{S}^2 = \nabla\left[\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot\nabla\left(\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\nabla\left((\boldsymbol{V}-\nabla H)\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\right)\right)\right],
$$
$$
\boldsymbol{S}^3 = \left(\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\nabla\right)(\boldsymbol{V}-\nabla H) - (V_i - \nabla_i H)\nabla\frac{(\operatorname{rot}\boldsymbol{V})_i}{\rho},
$$
$$
\boldsymbol{J}^0 = \frac{\operatorname{rot}\boldsymbol{V}}{\rho}, \quad \boldsymbol{J}^1 = \frac{1}{\rho}\operatorname{rot}\left[\left(\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\nabla\right)(\boldsymbol{V}-\nabla H) - (V_i-\nabla_i H)\nabla\frac{(\operatorname{rot}\boldsymbol{V})_i}{\rho}\right],
$$
$$
\boldsymbol{J}^2 = \tfrac{1}{2}[(\boldsymbol{V}-\nabla H)\times\boldsymbol{S}^k], \quad k = 1, 2, 3, \ldots.
$$

We do not cite more cumbersome equations, e.g. ${}^{(2)}\omega_S = i_J\mathrm{d}i_J\mathrm{d}i_J\mathrm{d}\omega_S$ and $\boldsymbol{J}''$.

These invariants (apart from I^0, $\boldsymbol{S}^0$ and $\boldsymbol{J}^0$) are higher ones since they include the derivatives of the field whose order exceeds those which appear in equations of motion.

The local dynamic invariants presented generate integral invariants in accordance with (5.1). Let us give some examples. The charge conservation law of frozen-in integrals (see (6.1)) yields

$$
I_1^q = \oint_{S(t)} (\boldsymbol{V}-\nabla H)\times\nabla\left(\frac{1}{\rho}(\boldsymbol{V}-\nabla H)\cdot\operatorname{rot}\boldsymbol{V}\right)\mathrm{d}\boldsymbol{s}',
$$
$$
I_2^q = \oint_{S(t)} (\boldsymbol{V}-\nabla H)\times\nabla\left[\frac{1}{\rho}\operatorname{rot}\boldsymbol{V}\cdot\nabla\left[\frac{1}{\rho}\operatorname{rot}\boldsymbol{V}\cdot(\boldsymbol{V}-\nabla H)\right]\right]\mathrm{d}\boldsymbol{s}',
$$
$$
I_3^q = \oint_{S(t)} (\boldsymbol{V}-\nabla H)\times\boldsymbol{S}^2\,\mathrm{d}\boldsymbol{s}'.
$$

The mass conservation law for the corresponding quantities leads to

$$
I_0^3 = \int f\rho\,\mathrm{d}x^3, \quad I_1^3 = \int f(\boldsymbol{V}-\nabla H)\frac{1}{\rho}\operatorname{rot}\boldsymbol{V}\,\mathrm{d}x^3,
$$
$$
I^3 = \int f\operatorname{rot}\boldsymbol{V}\cdot\nabla\left[\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\cdot\nabla\left[(\boldsymbol{V}-\nabla H)\cdot\frac{\operatorname{rot}\boldsymbol{V}}{\rho}\right]\right]\mathrm{d}x^3
$$

where f is an arbitrary function of Lagrangian invariants,

$$f = f\left(\left(\frac{1}{\rho}(V-\nabla H)\cdot \mathrm{rot}\, V\right), \frac{\mathrm{rot}\, V}{\rho}\cdot\nabla\left[(V-\nabla H)\cdot\frac{\mathrm{rot}\, V}{\rho}\right], \ldots\right).$$

One can easily construct topological invariants from (6.4) and (6.25) with the use of the local dynamic invariants obtained. For instance:

$$I_1^\tau = \int_{V(t)} (V-\nabla H)\cdot \mathrm{rot}\, V \mathrm{d}x^3,$$

$$I_2^\tau = \int_{V(t)} \mathrm{rot}\left[\left(\frac{\mathrm{rot}\, V}{\rho}\cdot\nabla\right)(V-\nabla H) - ((V-\nabla H)\cdot\nabla)\frac{\mathrm{rot}\, V}{\rho}\right] \cdot\left[\left(\frac{\mathrm{rot}\, V}{\rho}\nabla\right)(V-\nabla H) - ((V-\nabla H)\nabla)\frac{\mathrm{rot}\, V}{\rho}\right]\mathrm{d}x^3.$$

When the I^τ-type invariants vanish identically, which implies vanishing of the integrand, complementary topological invariants I^g arise. If, for example, $(V-\nabla H)\,\mathrm{rot}\, V = 0$ then

$$I_1^g = \int_{V(t)} \frac{[(V-\nabla H)\times \mathrm{rot}\, V]}{(V-\nabla H)^2}\mathrm{rot}\left[\frac{[(V-\nabla H)\times \mathrm{rot}\, V]}{(V-\nabla H)^2}\right]\mathrm{d}x^3.$$

Similarly, another complementary invariant I^g arises when $I_2^\tau = 0$.

The S-invariants of the gradient type do not generate topological invariants I^τ but, as in non-compressible fluid, they generate I^g invariants. For example, $S = I^0 S^1$ leads to topological invariants of the following type:

$$I^g = \int \frac{\nabla I^0}{(I^0)^2}\left(S^1\times\nabla\left(\frac{\nabla I^0\cdot S^1}{(S^1)^2}\right)\right)\mathrm{d}x^3.$$

An analogous invariant arises for $S = I^0 S^2$.

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